

A Multi-Portfolio Model for Bespoke CDO Pricing

Part I: Methodology

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Abstract

This paper presents a dynamic multi-portfolio default model for consistent and arbitrage-free pricing synthetic CDO tranches that reference a bespoke portfolio. In order to incorporate standard tranche price information, we assume that the bespoke portfolio has name overlapping with some index portfolios.

Dividing the total portfolio (parent) into non-overlapping sub-portfolios (children), and assuming homogeneity for both the parent and the children, we use a top-down dynamic default intensity model for the parent, and specify the conditional probability of default in the children given imminent default in the parent. We consider two fundamental cases which are building blocks of more complex applications: (a) the parent is an index and the bespoke is a child; and (b) the bespoke is the parent that contains one or more indices as children.

When the parent is the index, the parent default process is uniquely determined independent of the children, and the child conditional default probability distribution is calibrated to the spreads of the children. When the bespoke is the parent and one or more children are indexes, we simultaneously calibrate the parent default intensity model and the child default probability to the standard tranches and child portfolio spreads.

The model is designed to establish consistency between the pricing of standard tranches and the pricing of bespoke tranches. Application may include

- Portfolio enlargement where the bespoke tranche references a “global” portfolio that contains “regional” indexes as sub-portfolio. For example, tranches referencing CDX.NA.IG and iTraxx Europe.
- Portfolio thinning where the bespoke tranche references a sub-portfolio of an index.
- Combination of portfolio enlargement and thinning. For example, tranche referencing a subset of CDX and a subset of iTraxx.

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1 Introduction

Recent years have seen significant progress in the modeling of CDO tranches with the emergence of the top-down Markovian contagion, or self-affecting, dynamic credit default models. Top down refers to the modeling approach where the credits in the portfolio are assumed to have the same characteristics – notional amount, default probability, recovery rate – and hence only the number and timing of credit default are important to the calculation of the portfolio credit loss. The identity of the defaulter is irrelevant. In mathematical term, at a given time, the natural filtration at time t generated by the portfolio default history contains the information about number and timing of default up to t , but not the identities of the defaulters. Credit default contagion refers to the model feature that the portfolio default intensity jumps post default, reflecting the empirical observation that the surviving credits in the portfolio generally become more likely to default after a default.¹ In other words, credit default has contagious effect on the financial health of the firms still alive.

In the reduced-form credit default contagion models, the default counting process is modeled by a self-affecting intensity process that accounts for the empirical evidence that defaults are clustered where more credit defaults are likely to follow after a default. The fact that the default contagion models, whether they are default only in which spread volatility is due entirely to defaults [Herbertsson (2007)] or they account for stochastic spread volatility [Arnsdorf and Halperin (2007), Ding *et al* (2008), Giesecke and Goldberg (2005), Lopatin and Misirpashaev (2007)], can achieve almost perfect calibration to the standard index tranches suggests that the default contagion is a major risk factor of the credit portfolio products, and any sensible portfolio default model must account for this effect in some way.

In addition to the near perfect calibration to the standard tranches, bespoke tranches with non-standard strikes and maturities can be priced and risk managed consistently using the same model. As a result, there is no longer the need to use the *ad hoc* base correlation interpolation/extrapolation schemes, thereby eliminating a major model inconsistency that is inherent in the current market standard one-factor copula pricing model [Morgan and Mortensen (2007), Schloegl et al (2008)].

However, pricing and risk managing of tranches referencing bespoke portfolio still pose a tremendous challenge due to the illiquid nature of the bespoke portfolio. Complete model calibration to bespoke portfolio is difficult, if not impossible, because of lack of prices on tranches referencing the bespoke portfolio. The current market standard for bespoke tranche pricing is the one-factor Gaussian copula model with some form of base correlation mapping scheme between the base correlation of the standard tranche and that of the bespoke tranche [Baheti and Morgan (2007), Turc *et al* (2006)]. Although these mapping schemes have some theoretical justifications, they are *ad hoc* in nature and known to admit arbitrage [Morgan and Mortensen (2007), Schloegl et al (2008)].

¹ Strictly speaking, surviving firms do not necessarily all become riskier after a default. For example, a firm may be stronger after a default by its competitor. But the systemic risk generally increases after default.

Furthermore, the base correlation mapping is static which makes risk management of bespoke tranche problematic because different tranches of the same portfolio will have to be valued and hedged using different mapping. This problem is akin to hedging an equity option portfolio using different implied volatility. It is thus desirable to develop dynamic and consistent models that incorporate available index pricing information into pricing of bespoke tranche products.

In this paper, we present a dynamic multi-portfolio default model for the pricing of tranche referencing bespoke portfolio. The purpose is to make use of the pricing information on indexes² that overlap with the bespoke portfolio such that the bespoke tranche is priced consistently with the indexes. To this end, we make a fundamental assumption that the bespoke portfolio has the child-parent relationship with an index or several indexes. A parent is defined as the portfolio that contains all credits. A child is a sub-portfolio consisting of a portion of the credits in the parent. The bespoke portfolio can be either the parent or a child. We refer the former as (random) portfolio enlargement in the sense that the bespoke portfolio is an expansion of the indexes, and the latter as (random) portfolio thinning where the bespoke portfolio is a contraction of an index. The portfolio thinning is similar to the random thinning to single name CDS proposed by Giesecke and Goldberg (2005), and includes the random thinning to single names as a special case when the children contain only one credit. In the portfolio enlargement, one of the children must be an index portfolio. In the portfolio thinning, the parent must be an index. More complex cases where the bespoke portfolio partially overlaps with the index can be handled using a combination of portfolio enlargement and thinning.

Through the child-parent relationship between the bespoke portfolio and the index, the standard tranche price information is incorporated into the bespoke portfolio default model and hence into the bespoke tranche pricing. This is another way of achieving the same goal as that of base correlation mapping between the standard tranche and bespoke tranche. While the motivation behind our model is the same as that of base correlation mapping, there are fundamental differences. First, the base correlation mapping is static in the sense that it applies only to each individual tranche, namely, the base correlation mapping is maturity, strike and reference portfolio specific. Different tranches need different mapping, creating inconsistency which is especially problematic for risk management. This consistency problem is similar to risk managing equity option portfolio using Black-Scholes model and Black implied volatility, where options of different expiries and strikes are valued using different implied volatilities making portfolio hedging problematic. Second, the base correlation mapping is not arbitrage free. Our method is dynamic and once the model is calibrated, it values bespoke tranches in a consistent and arbitrage-free manner, across all maturities and strikes.

The rest of the article is organized as follows. In section 2, we first define some symbols that will be used through out this paper. We then describe the default processes for the parent and the child portfolios. We assume the parent default process is given, and focus on the child portfolio conditional default probability distribution using the fact that child portfolio default process is derived based on the parent default process and the

² Index is referred to as a reference portfolio of liquid tranches. Examples of index are CDX and iTraxx.

conditional default probability of child portfolio. Section 3 outlines the tranche and index pricing methodologies that will be used in fitting the model to the market prices of the index and standard tranches. Section 4 gives the calibration procedures. If the parent portfolio is the index, we calculate the parent default process intensity independently of the child process by fitting to the index and standard tranches. We then calculate the child portfolio conditional default probability by fitting the child default process, which contains only the conditional default probability as unknown, to the child portfolio average spread and other available liquid prices. If the child portfolio is the index and the parent portfolio is bespoke, it is necessary to jointly calibrate the parent default process and the child portfolio conditional default probability. Section 5 discusses potential applications of the model, and section 6 concludes the paper.

2 Model Description

2.1 Nomenclature

The following notations are used throughout this paper.

- $\tau_1^P < \tau_2^P < \dots < \tau_N^P$: Ordered default times in the parent which contains N obligors.
- $\tau_1^k < \tau_2^k < \dots < \tau_{M_k}^k$: Ordered default times in child k which contains M_k obligors.
- $T_0 < T_1 < T_2 < \dots < T_k$: Coupon payment date set.
- $\lambda(n, t) = \lambda(N_t^P = n, t)$: The next-to-default (NTD) intensity of the parent.
- $\tilde{F}_t^P = \sigma\{\tau_k^P \leq t\} \cup \{\lambda_s \mid s \leq t\}$: The natural filtration generated up to time t by the default in the parent portfolio and the parent default intensity.
- $\tilde{F}_t^k = \sigma\{\tau_j^k \leq t\}$: The natural filtration generated by the default in the child k .
- $N_t^P = \text{Sup}\{k \mid \tau_k^P \leq t\}$: The default level in the parent.
- $\vec{N}_t^C = \{N_t^1, \dots, N_t^K\} = \vec{m} = \{m_1, \dots, m_K\}, m_k = \text{Sup}\{j \mid \tau_j^k \leq t\}, k = 1, \dots, K$: Vector of default level state of the children. Note that in our model setting, $N_t^P = n = \sum_{k=1}^K m_k$
- $P(n, \vec{m}, \lambda_t, t) = P(N_t^P = n, \vec{N}_t^C = \vec{m}, \lambda_t, t)$: The joint probability density of stochastic intensity process with \vec{N}_t^C defaults among the children and at intensity λ_t .
- $P(n, \vec{m}, t) = P(N_t^P = n, \vec{N}_t^C = \vec{m}, t)$: The joint probability density of n defaults in the parent portfolio and $\vec{m} = (m_1, \dots, m_K)$ defaults among the children.
- $\mu^k(m_k, t)$: The NTD intensity in child k conditional on m_k prior defaults.
- $D(T) = \text{Exp}\left\{-\int_0^T r(s)ds\right\}$: The risk-free discount factor from T .

2.2 Model Framework

The model is designed to provide a dynamic, consistent, and arbitrage free valuation of tranches referencing a bespoke portfolio. To this end, we assume that the reference bespoke portfolio has a child-parent relation with one or more index portfolios. The indices are assumed to have no name overlapping among each other. We call the overall portfolio that contains all the credits the parent portfolio or simply the parent, and the sub-portfolios children.

We consider two fundamental cases. In the first case, the parent is the bespoke portfolio that contains one or more index portfolios as sub-portfolios, perhaps plus the remaining names not belonging to any of the indices. In this case, the parent and the children are determined by the bespoke portfolio and the available indices. For example, if we want to price tranches referencing names in CDX and iTraxx, the parent contains the 125 names in the CDX index and the 125 names in the iTraxx index. The two natural children are the CDX and the iTraxx. In the second case, the bespoke portfolio is a sub-portfolio of an index which is the parent. More complicated case can be constructed based on combining these two.

Suppose the parent contains N obligors, each with constant recovery rate R and notional 1. Each default then results in a loss of $(1 - R)$. While the assumption of constant notional is required by the top-down model, the constant recovery is not necessary but it affords great convenience for exposition and computational efficiency.

Let the parent be divided into K non-overlapping children, we regard both the parent and the children to be homogeneous. The recovery rate and notional amount of a credit are the same whether the credit is viewed as a member of the parent or a member of the child to which the credit belongs. However, the spread of a credit when viewed as a member of the parent is different from that when viewed as a member of the child. The spread when viewed from the parent is equal to the spread of the parent, and that when viewed from the child is equal to the spread of the child. Since the child contains only a sub-set of the parent, the spread of the child differs from that of the parent.

The above model setting assumes that the default is driven by the parent default intensity, and the default in the parent is distributed among the children. Therefore, our model is comprised of a default intensity process for the parent and a distribution of conditional probability of default among the children.

Remark 2.2.1: For simplicity, we assume a constant recovery rate. We recognize that a stochastic recovery rate or a time dependent deterministic recover rate may be required to achieve better fitting to the market prices when the market is under stress. Recent “credit crunch” has resulted in significant increase in the prices of the senior and super-senior index tranches. For example, the 60-100% tranche, which has no value under the customary assumption of 40% recovery, was recently trading with significant value [22]. Our model framework can easily incorporate non-constant recovery rate assumption at the parent portfolio level.

2.3 The Parent Default Intensity Process

Assuming homogeneity, the default intensity process of the parent portfolio in a top-down Markovian contagion default model can be generally expressed as³

$$d\lambda_t = \mu(\lambda_t, N_t^P, t)dt + \sigma(\lambda_t, N_t^P, t)dW_t + \xi(\lambda_t, N_t^P, t)dN_t^P \quad (1)$$

where dW_t is the standard Weiner process and N_t^P is the parent portfolio default level defined as

$$N_t^P = \sum_{k=1}^N 1(\tau_k^P \leq t) \Rightarrow dN_t^P = N_t^P - N_{t-}^P = \sum_{k=1}^N [1(\tau_k^P \leq t) - 1(\tau_k^P < t)] \quad (2)$$

where $1(x)$ is the indicator function and τ_k^P is the k-th default time in the parent portfolio.

There are a number of top-down Markovian contagion models in the literature (See references [1], [4], [6], [12], [16] and the references thereafter). Below, we mention four such models, representing distinct model features, numerical methods, and practical applications.

2.3.1 Affine Intensity Model [6]

Errais *et al* [6] proposed a class of affine intensity models with general formalization. In this paper, we mention one particular affine models which is formalized as⁴

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + \delta\xi dN_t^P \quad (3)$$

where $\kappa, \lambda_\infty, \sigma, \delta$ are constant and ξ is the loss-given-default (LGD) that can be defined either as a random variable following an independent distribution or as a constant.

A stochastic LGD provides an additional parameter that generally results in a better fit to the market tranche prices at the expense of introducing another dimension. ξ can also be specified as a function of default time τ_k^P to introduce more parameters for potentially better fitting. One simple specification of random recovery is the uniform distribution [6].

Affine model defined in Eq. (3) is self-affecting where the last term in Eq. (3) causes the intensity to jump upon a new default, making future default more likely. When there is no further default, the intensity diffuses and mean-reverts to the long term equilibrium level

³ The BSLP model and the zero-volatility intensity model directly specify λ_t .

⁴ The affine models described in [6] are much richer than that of Eq. (3). Here, we use the model (3) for exposition purpose because the emphasis of this paper is on the multi-portfolio framework rather than the particular form of the parent default model.

λ_∞ . Between defaults, the affine intensity model is similar to the CIR interest rate model which guarantees that the intensity can never become negative.

Affine intensity model (3) can be solved either by Monte Carlo simulation method [8] or by the characteristic function method [6][7] adapted to the multi-portfolio setting. Monte Carlo method for pricing under the multi-portfolio setting requires a two-step simulation. In the first step, the parent default is simulated where the distribution is governed by Eq. (3). The algorithm of Giesecke and Kim (2007) can be used in the first step. If a default is encountered, the second step simulation, which can be considered as an inner simulation, determines which child the defaulter belongs. The distribution of the second step is governed by the conditional multi-portfolio default probability density $C_k(n, \vec{m}, t)$. Monte Carlo simulation generally requires daily time step to account for the possibility of default clustering (default in consecutive days).

2.3.2 The Lopatin and Misirpashaev Model [16]

Assuming constant recovery rate across the parent, the Lopatin and Misirpashaev model [16] is parameterized as

$$d\lambda_t = \kappa(\rho(N_t^P, t) - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t \quad (4)$$

Instead of feeding the default back to the intensity process through the explicit linear term dN_t^P as done in the affine model, the credit default contagion effect is achieved by making the mean-reversion level $\rho(N_t^P, t)$ depending on the parent default level N_t^P . The higher the default level is, the higher the mean-reversion level becomes which forces the spread to reach a higher equilibrium level, and hence making default more likely. The diffusion term produces credit spread volatility between defaults and provides the extra degrees of freedom for pricing dynamic products.

This model does not have a transform based solution due to the mean-reversion function. It can be solved by solving the Kolmogorov forward equation (15) for the joint default level density function $P(N_t^P, \vec{N}_t^C, \lambda, t)$ which is defined below. Since Eq. (15) is a multi-portfolio extension of Eq. (2) of [16] for single-portfolio, the numerical scheme in [16] can be adapted to solve Eq. (15). The Monte Carlo method outlined in the previous above for the affine model can be used for this model.

2.3.3 The BSLP Model [1]

The intensity of the BSLP model proposed by Arnsdorf and Halperin (2007) is more conveniently written as

$$\lambda(N_t^P, Y_t, t) = (N - N_t^P)F(N_t^P, t)Y_t \quad (5)$$

where the contagion factor $F(N_t^P, t)$ is a deterministic bilinear function of the default level and time. The random factor Y_t accounts for the spread volatility between defaults, and is assumed to follow a mean-reversion jump-diffusion model. Furthermore, Y_t is assumed to be dependent on dN_t^P but not the default level N_t^P itself which enables to calculate the conditional probability $P(Y_{t+\Delta t} | N_{t+\Delta t}^P, N_t^P, Y_t)$ using a large time step Δt . In the BSLP model, the burden of calibration and simulation of the default intensity λ_t is shifted to the stochastic factor Y_t which is usually much simpler than λ_t . The BSLP model can be further simplified if Y_t is independent of the default level and is only a diffusion process in which case the diffusion process Y_t and the default level N_t^P are decoupled. The BSLP model can be solved by tree method with large time step [1].

2.3.4 Zero-Volatility Intensity Model [12]

The zero-volatility intensity model is obtained by letting $Y_t \equiv 1$ in Eq. (5), so it can be considered as a special case of the BSLP model. It is also called default only model since the random intensity change comes entirely from the jump in default level. Herbertsson (2007) considered a zero-volatility intensity model in which the intensity is specified as

$$\lambda_t(N_t^P, t) = (N - N_t^P) \left(\frac{\lambda_0}{N} + \sum_{k=1}^{N_t^P} \xi(k, \tau_k^P) \right) \quad (6)$$

where $\xi(k, t)$ is a piecewise constant function of k and t ,

$$\xi(k, t) = \xi_{i,j}, \text{ where } k \in (k_{i-1}, k_i], t \in (T_{j-1}, T_j] \quad (7)$$

In practice, the choices of the spatial node k_i and the temporal nodes T_j are problem dependent and may be less obvious than the single-portfolio case. In the single-portfolio case, k_i and T_j are usually chosen to coincide with the detachment points and maturity of the standard tranches. For example, we have $\{k_i\}_{iTraxx} = \{0\%, 3\%, 6\%, 9\%, 12\%, 22\%\}$, and $\{k_i\}_{CDX} = \{0\%, 3\%, 7\%, 10\%, 15\%, 30\%\}$. It is less obvious when the bespoke portfolio contains both iTraxx and CDX because the detachment points for iTraxx tranches and CDX tranches are different.

Since $\xi(k, t)$ is positive, the intensity $\lambda(N_t^P, t)$ is a non-decreasing function of N_t^P , accounting for the default contagion effect. It is easy to see from Eq. (6) that the credit spread is constant between defaults. Herbertsson shows that this model is capable of perfectly reproducing the market prices of the standard tranches. A distinct advantage of this model is that it enables analytic tranche pricing formula due to the constant Markov generator matrix.

The zero-volatility is sufficient for tranche pricing. Its numerical implementation is the simplest among the four models described above. For single- [12] and two-portfolio cases, matrix exponential method can be used. The two-portfolio case solution is presented in Appendix. For practical purpose, the number of children is likely limited to three with the bespoke portfolio being one of the children.

Below, we will demonstrate that the zero-volatility default model is not suitable for pricing spread volatility sensitive products such as option on tranche and forward tranche because these products require sampling multiple points in time, and hence termed dynamic products.

Finally, we note that any of the above models can be used as the parent default intensity model. The choice depends on the model behavior and the ease of implementation and computation efficiency.

2.4 The Child Portfolio Conditional Default Probability

In this section, we describe the conditional default probabilities of the child portfolios, given the default process for the parent portfolio.

Suppose the parent is divided into K non-overlapping children, and child k has M_k initial obligors. Let default state of the children be denoted by $\vec{N}_t^C = \vec{m} = (m_1, \dots, m_K)$ where m_k is the default level in child k . It is clear that the parent default level must be equal to the sum of all children's default levels,

$$n = \sum_{k=1}^K m_k \quad \text{and} \quad 0 \leq m_k \leq M_k \quad (8)$$

Define the probability that the next default is in child k conditional on the current default state \vec{m} by

$$C_k(n, \vec{m}, t) = P(\tau_{n+1}^P = \tau_{m_k+1}^k \mid \tau_n^P \leq t, \tau_{m_k}^k \leq t, \tilde{F}_t^P \cup (\cup \tilde{F}_t^j)) \quad (9)$$

then the child conditional default probability must satisfy

$$C_k(n, \vec{m}, t) \geq 0 \quad \text{and} \quad \sum_{k=1}^K C_k(n, \vec{m}, t) = 1 \quad (10)$$

The explicit dependency on \vec{m} emphasizes that the conditional default probability of child k depends on the entire default state of all children, not just the default level of child k . Intuitively, we would expect that, given the default count m_k in the child k and the total defaults in the parent, more defaults in children having higher spreads than child k

and hence less defaults in less risky children makes it more likely that the next default will be in child k .

It is obvious that when $m_k = M_k$, all credits in child k have defaulted. The next default cannot be in child k . So the probability that next default is in child k is zero. On the other hand, if all credits in children other than child k have defaulted, the next defaulter must be in child k . When all surviving credits are in child k , we have $N - n = M_k - m_k$. Therefore, the conditional default probability must also satisfy

$$C_k(n, \vec{m}, t)|_{m_k=M_k} = 0 \quad \text{and} \quad C_k(n, \vec{m}, t)|_{m_k=M_k-N+n} = 1 \quad (11)$$

A specification of child conditional default probability that satisfies constraints (10) and (11) is

$$C_k(n, \vec{m}, t) = \frac{x_k g_k(x_k)}{\sum_{j=1}^K x_j g_j(x_j)} \quad (12)$$

where

$$x_k = (M_k - m_k)/(N - n) \quad (13)$$

$g_k(x)$ is some positive, deterministic function satisfying $g_k(1) = 1$. $x_k = 1$ is equivalent to $N - n = M_k - m_k$ and $x_k = 0$ corresponds to $m_k = M_k$. For model consistency and symmetry, namely, all children must be treated the same way, $g_k(x)$ should have the same functional form for all k . However, the parameters of $g_k(x)$ can be different to reflect the difference among the average spreads of the children. In other words, we can specify $g_k(x) = g(x; \xi_k)$ where ξ_k is a (time-dependent) parameter (potentially a vector) specifically for child k .

Eqs. (12) and (13) are a general specification of the child conditional default probability that must satisfy the conditions (10) and (11). However, we still have freedom to choose function $g(x; \xi)$ which can conceivably depend on application. The simplest form for $g_k(x)$ is a linear function.

$$g_k(x_k) = \xi_k(t) + x_k [1 - \xi_k(t)] \quad (14)$$

where $\xi_k(t)$ is a deterministic function of time. In practice, $\xi_k(t)$ is either piecewise constant or piecewise linear and must be fitted to the market data.

Although the specification (14) is the simplest function form under the framework of formulas (12) and (13), it nonetheless has some interesting features.

First, since $m_k \leq M_k$ and $n = |\bar{m}|$, we see, by virtue of Eq. (13), that $x_k \leq 1$ for all $k \leq K$. Furthermore, if a default occurs in the child k , then x_k decreases and $x_{j \neq k}$ increases.

Second, $g_k(x_k)$ is an increasing function of the parameter ξ_k where ξ_k represents the (average) default risk level of the child k relative to the (average) default risk level of the parent. If $\xi = 1$ for a child whose spread is equal to the spread of the parent, then we have $\xi_k(t) > 1$ if the spread of the child k is higher than the parent spread, and $\xi_k(t) < 1$ if the spread of the child k is lower than the parent spread. This is desirable that more risky children have higher conditional default probabilities, other things being equal. Note that we can always scale

Third, $g_k(x_k)$ is an increasing function of x_k if $\xi_k(t) < 1$ and is a decreasing function of x_k if $\xi_k(t) > 1$. However, the conditional child default probability is given by (12) so the monotonicity on $g_k(x_k)$ is not shared by $C_k(n, \bar{m}, t)$. Numerical experiments confirm that a default by the child k results in a decrease in $C_k(n, \bar{m}, t)$ and an increase in $C_j(n, \bar{m}, t)$ for all $j \neq k$. This indicates that when child k experiences a default, the likelihood that the next default is also in child k decreases. Since the sum of the child conditional default probabilities must be equal to 1, the probability that the next default being in other child than k is likely to increase. In other words, the linear model (14) implies conditional anti-default clustering among the children. However, it is important to distinguish this feature from the default clustering implied by the parent default contagion model.

Finally, it is important to note that $g_k(x_k), k = 1, \dots, K$ must be calibrated simultaneously to the market data. In the non-trivial case, the parent portfolio is divided into two or more sub-portfolios. This means that $\xi_k(t)$ are determined together.

Remark 2.4.1: If the children are identical in the sense that they have the same intrinsic spread and contain the same number of credits, $C_k(n, \bar{m}, t)$ must be the same across k because in this case one child cannot be riskier than another. By virtue of Eqs. (12), (13) and (14), this is equivalent to constant $\xi_k(t)$ across k . Setting $\xi_k(t) = \xi$ in Eq. (14), we conclude that the only possibility is $\xi = 1$ and $C_k(n, \bar{m}, t) = x_k$.

Remark 2.4.2: Single name default dynamics can be obtained by setting $M_k = 1$. In this case, we have $x_k = (1 - 1(\tau^k < t)) / (N - n)$ where τ^k is the default time of child k which is a single credit. When the k -th credit defaults, $m_k = 1$ and $C_k(n, \bar{m}, t) = 0$. Consequently, we have $\mu_k^{NTD}(m_k = 1, t) = 0$. In other words, the single name default intensity drops to zero after default. However, we now need to track the defaulter identity through the natural filtration $\tilde{F}_t^k = \sigma(\tau^k \leq t)$ for child k .

Remark 2.4.3: When there is only one child portfolio, i.e. the child is the same as the parent, we have $m_1 = n$ and $M_1 = N$. This results in $x_1 = 1$ and $C_k \equiv 1$. Hence, in this case the child default dynamics automatically degenerates to the parent process.

Remark 2.4.4: The dependence of $C_k(n, \bar{m}, t)$ on n is technically redundant because of the constraint (8). Given the child default level state $\bar{N}_t^C = \bar{m}$, the parent default level $N_t^P = n = |\bar{m}|$ is uniquely determined. Similarly, the unconditional multi-portfolio default probability density $P(n, \bar{m}, \lambda, t)$ is uniquely determined by \bar{m} . Nevertheless, we will keep the N_t^P dependence explicit to emphasize the parent default level.

Remark 2.4.5: Eqs. (12) and (13) are a general form of the child conditional default probability based on the symmetry and boundary conditions. We still have flexibility to choose that affords flexibility the linear form of Eq. (14) is the simplest specification

2.5 Joint Multi-Portfolio Default Probability Density

The joint unconditional default probability density

$P(n, \bar{m}, \lambda, t) = P(N_t^P = n, \bar{N}_t^C = \bar{m}, \lambda_t = \lambda, t)$ of the default counting process (N_t^P, \bar{N}_t^C) , with the parent intensity defined by Eq. (1) and the child conditional default probability $C_k(N_t^P = n, \bar{N}_t^C = \bar{m})$ satisfies the Kolmogorov forward equation adapted to our multi-portfolio stochastic intensity process

$$\begin{aligned} \frac{\partial P(n, \bar{m}, \lambda, t)}{\partial t} = & -\frac{\partial}{\partial \lambda} [\mu(\lambda, n, t)P(n, \bar{m}, \lambda, t)] + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [\sigma^2(\lambda, n, t)P(n, \bar{m}, \lambda, t)] \\ & + \lambda \sum_{k=1}^K P(n-1, \bar{l}_k, \lambda, t) C_k(n-1, \bar{l}_k, t) - \lambda P(n, \bar{m}, \lambda, t) \end{aligned} \quad (15)$$

where $\bar{l}_k = \{l_{j \neq k} = m_j, l_k = m_k - 1\}$ is the child default state that can reach the state \bar{m} in an infinitesimal time interval. Note that the joint default probability density is non-zero only if $n = |\bar{m}|$ and $n-1 = |\bar{l}_k|$.

If the default intensity is zero, then there is no chance of default. Therefore, we impose the boundary condition

$$P(n, \bar{m}, 0, t) = 0 \quad (16)$$

Obviously, there is no default at time $t = 0$. Therefore, the initial condition for the default density function for the stochastic intensity is

$$P(n, \bar{m}, \lambda, 0) = f(\lambda) \times 1(n = 0) \quad (17)$$

By definition, the stochastic default intensity for the parent as a single portfolio is

$$P(N_t^P = n, \lambda, t) = \sum_{|\vec{m}|=n} P(n, \vec{m}, \lambda, t) \quad (18)$$

where the summation is over all possible state \vec{m} satisfying $|\vec{m}| = \sum_{k=1}^K m_k = n$.

Because the default state \vec{m} can be reached only from the state $\vec{l}_k, k = 1, \dots, K$, over an infinitesimal time interval, we have

$$\sum_{|\vec{m}|=n} \sum_{k=1}^K P(n-1, \vec{l}_k, \lambda, t) C_k(n-1, \vec{l}_k, t) = P(n-1, \lambda, t) \quad (19)$$

Substituting Eqs. (17), (18) and (19) into Eq. (15), we obtain the Kolmogorov forward equation for the stochastic parent default density

$$\begin{aligned} \frac{\partial P(n, \lambda, t)}{\partial t} = & -\frac{\partial}{\partial \lambda} [\mu(\lambda, n, t) P(n, \lambda, t)] + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [\sigma^2(\lambda, n, t) P(n, \lambda, t)] \\ & + \lambda [P(n-1, \lambda, t) - P(n, \lambda, t)] \end{aligned} \quad (20)$$

Note that Eq. (20) is the same as the Eq. (2) in [16].

Pricing of credit derivatives entails the calculation of expectation of some payoffs which, at present, are generally functions of portfolio default loss but not the intensity. Assuming constant recovery, loss is proportional to the default level. As a result, the payoff is a function of the portfolio default level. The model calibration involves fitting to the prices of the standard tranche and index. The model may also need to be calibrated to option prices to determine the dynamic model parameters controlling the spread volatility and mean reversion.

Calibration to the current market prices of standard tranches and index requires only the probability density of joint default state among the children which is independent of λ_t . Thus, it is advantageous to eliminate the λ_t dependence in this part of the calibration. To this end, we first note that the joint probability satisfies the following transition equation

$$\begin{aligned} P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C, \lambda_{t+\Delta t}, t + \Delta t) = & \sum_{N_t^P, \vec{N}_t^C} \int_0^\infty P(N_t^P, \vec{N}_t^C, \lambda, t) \times \\ & P(\lambda_{t+\Delta t} | N_{t+\Delta t}^P, N_t^P, \lambda) P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C | N_t^P, \vec{N}_t^C, \lambda) d\lambda \end{aligned} \quad (21)$$

where we have used the fact that the parent default intensity λ_t is independent of the child default level vector \vec{N}_t^C .

Integrating both sides of Eq. (21) with respect to $\lambda_{t+\Delta t}$, we obtain the unconditional joint multi-portfolio probability density of $\bar{N}_{t+\Delta t}^C$ at time $t + \Delta t$,

$$P\left(N_{t+\Delta t}^P, \bar{N}_{t+\Delta t}^C, t + \Delta t\right) = \sum_{N_t^P, \bar{N}_t^C} \int_0^\infty P\left(N_t^P, \bar{N}_t^C, \lambda, t\right) P\left(N_{t+\Delta t}^P, \bar{N}_{t+\Delta t}^C \mid N_t^P, \bar{N}_t^C, \lambda\right) d\lambda \quad (22)$$

where the unconditional joint default probability density is defined by

$$P\left(N_t^P, \bar{N}_t^C, t\right) = \int_0^\infty P\left(N_t^P, \bar{N}_t^C, \lambda, t\right) d\lambda \quad (23)$$

Making the usual assumption of no simultaneous defaults, the conditional probability of next defaulter being in child k is

$$P\left(N_{t+\Delta t}^P = n, \bar{N}_{t+\Delta t}^C = \bar{m} \mid N_t^P = n - 1, \bar{N}_t^C = \bar{l}_k, \lambda_t\right) = \lambda_t (n - 1) \Delta t C_k \left(n - 1, \bar{l}_k\right) \quad (24)$$

Substituting (24) into (21), setting $N_{t+\Delta t}^P = n, \bar{N}_{t+\Delta t}^C = \bar{m}$, and using the relation (note that λ_t is independent of the child default level \bar{N}_t^C),

$$\int_0^\infty \lambda P\left(N_t^P, \bar{N}_t^C, \lambda, t\right) d\lambda = P\left(N_t^P, \bar{N}_t^C, t\right) E\left(\lambda_t \mid N_t^P\right) \quad (25)$$

we obtain the forward differential equation for the unconditional joint child default probability density $P(n, \bar{m}, t)$ ⁵ (dependence on n is redundant and only for clarity)

$$\frac{dP(n, \bar{m}, t)}{dt} = -G(n, t)P(n, \bar{m}, t) + G(n - 1, t) \sum_{k=1}^K C_k \left(n - 1, \bar{l}_k, t\right) P(n - 1, \bar{l}_k, t) \quad (26)$$

where $\bar{l}_k = \{l_{j \neq k} = m_j, l_k = m_k - 1\}$ and

$$G(n, t) = E\left(\lambda_t \mid N_t^P = n\right) = \int_0^\infty \lambda P\left(N_t^P = n, \lambda, t\right) d\lambda / P\left(N_t^P = n, t\right) \quad (27)$$

$$P\left(N_t^P = n, t\right) = \int_0^\infty P\left(N_t^P = n, \lambda, t\right) d\lambda$$

The initial condition for Eq. (26) that is consistent with Eq. (17) is

$$P(n, \bar{m}, 0) = \int_0^\infty f(\lambda) d\lambda \times 1(n = 0) = \delta_{n,0} \quad (28)$$

⁵ Eq. (26) can also be derived by integrating Eq. (15) with respect to λ and by using Eq. (27).

It can be shown that the unconditional parent default density satisfies the equation

$$\frac{dP(n,t)}{dt} = -G(n,t)P(n,t) + G(n-1,t)P(n-1,t) \quad (29)$$

where $P(n,t) = \sum_{|\vec{m}|=n} P(n,\vec{m},t)$ with $|\vec{m}| = \sum_{k=1}^K m_k$. Eq. (29) is the same as Eq. (6) in [16].

Lopatin and Misirpashaev (2007) called $G(n,t)$ the local intensity (LI) because it is the expected intensity conditional on the default level. They extend the method of Markovian projection, a term used first used by Piterbarg (2007) for fast calibration of stochastic volatility interest rate model but the method was used earlier by Dupire (1994) for the local volatility model for equity derivative, to the credit derivative modeling. If the stochastic intensity λ_t is self-affecting, a higher level of default will on average result in a higher default intensity. Therefore, the local intensity $G(n,t)$ will be an increasing function of n under the default contagion models.

Remark 2.5.1: By deriving the local intensity model equation (26), we have effectively proven that $G(N_t^P, t)$ is the local intensity of the original stochastic intensity model, in the sense that the local intensity is the conditional expectation of the stochastic intensity.

Remark 2.5.2: The local intensity $G(n,t)$ is implicitly conditioned on today's filtration \tilde{F}_0^P . The Gyongy theorem, extended to jump process, implies that the SI model and the LI model have the same marginal density conditional on today's information, and both models are equivalent for pricing static products. But they generally differ for pricing dynamic products that depend on future filtration. We will elaborate in Sec. 2.8.

Remark 2.5.3: If there is only one child, then the child must be identical to the parent. In this case, we must have $m = n$, $C(n,m,t) = 1$ and $P(n,m,t) = P(n,t)$. Eqs. (15) and (26) reduce to the Kolmogorov forward equation under the single-portfolio setting.

2.6 The Local Intensity $G(N_t^P, t)$

Eq. (26) indicates that the initial condition for the local intensity default density $P(n,\vec{m},t)$ must be consistent with the initial condition Eq. (17) for the stochastic intensity default probability density. On the one hand, the definition $G(n,t) = E(\lambda_t | N_t^P = n)$ suggests that $G(n > 0, t = 0) = 0$ since $P(n > 0, 0) = 0$. On the other hand, if we take $G(n,0)$ as the limit $G(n > 0, 0) = \lim_{t \rightarrow 0} G(n > 0, t)$, we will see below that it not possible for $G(n > 0, t = 0) = 0$.

By definition,

$$G(n = 0, 0) = \lim_{t \rightarrow 0} \int_0^\infty \lambda P(n = 0, \lambda, t) d\lambda / P(n = 0, t) = \int_0^\infty \lambda f(\lambda) d\lambda / \int_0^\infty f(\lambda) d\lambda \quad (30)$$

If $f(\lambda) = \delta(\lambda - \lambda_0)$, then $G(0,0) = \lambda_0$. On the other hand, the zero-volatility model leads to $G(n=0, t=0) = E(\lambda_0 | N_0^P = 0) = \lambda_0$, the initial stochastic intensity.

The small time asymptotic behavior of $P(N_t^P, \lambda_t, t)$ is dominated by the jump behavior. Neglecting the diffusion effect, Eq. (20) becomes

$$\frac{dP(n, \lambda, t)}{dt} = \lambda P(n-1, \lambda, t) - \lambda P(n, \lambda, t), \quad P(n, \lambda, 0) = f(\lambda) \mathbb{1}(n=0) \quad (31)$$

Thus the small time approximate solution is

$$P(n, \lambda, t) = f(\lambda) \times (\lambda t)^n e^{-\lambda t} / n! \quad (32)$$

This leads to the initial condition for the local intensity function

$$G(n,0) = \lim_{t \rightarrow 0} \int_0^\infty \lambda P(n, \lambda, t) d\lambda / P(n,t) = \int_0^\infty \lambda^{n+1} f(\lambda) d\lambda / \int_0^\infty \lambda^n f(\lambda) d\lambda \quad (33)$$

Eq. (33) shows that $G(n > 0, 0) \neq 0$.

The actual computation of the local intensity function $G(n, t)$ depends on the parent default model. In the following, we summarize the calculation of local intensity function under the four models described in Sec. 2.3.

2.6.1 The Lopatin and Misirpashaev Model

If λ_t is given by the Lopatin and Misirpashaev model Eq. (4), then

$$E(\lambda_t | N_t^P) = \Lambda(N_t^P, t) \quad (34)$$

which is the Eq. (8) in [16]. The calculation of the local intensity function $\Lambda(N_t^P, t)$ follows the procedure described in [16]. It is clear that Eq. (26) is the multi-portfolio extension of the local intensity equation for single portfolio [16].

2.6.2 Affine Model

In the affine intensity model (3), we can use the Fourier transform method (Giesecke (2007), Ding *et al* (2008)). Define the joint characteristic function,

$$\Phi(u, \Psi(v), t) = E(e^{iu\lambda_t + ivN_t^P}) = \sum_{k=0}^N E(e^{iu\lambda_t} | N_t^P = k) [1 - \Psi(v)]^k P(N_t^P = k) \quad (35)$$

where $\Psi(v) = 1 - e^{iv}$, we deduce that

$$E(\lambda_t | N_t^P = n)P(N_t^P = n) = -i \left\{ \frac{\partial^{n+1} \Phi(u, w)}{\partial u \partial w^n} \Big|_{u=0, w=1} \right\} \div \left\{ \frac{\partial^n \Phi(u, w)}{\partial w^n} \Big|_{u=0, w=1} \right\} \quad (36)$$

provided that the partial derivatives exist.

The Fourier transform method can be applied to any intensity model for which the characteristic function and its derivatives exist.

2.6.3 The BSLP Model

Under the BSLP model framework (5),

$$E(\lambda_t | N_t^P) = F(N_t^P, t) E(Y_t | N_t^P) \quad (37)$$

So the burden of calculating $E(\lambda_t | N_t^P)$ is shifted to calculating $E(Y_t | N_t^P)$. If Y_t is independent of the default level increment, i.e. $\gamma(Y_t, t) = 0$ in Eq. (18) of [1], Eq. (37) essentially reduces to calculating $E(Y_t | N_t^P) = E(Y_t)$. In practice, $E(Y_t)$ is much simpler to compute than $E(Y_t | N_t^P)$.

2.6.4 Zero-Volatility Intensity Model

As described in Sect. 2.3.4, the zero-volatility intensity is a deterministic function of the default level and time. Hence, we have

$$E(\lambda_t | N_t^P) = \lambda(N_t^P, t) = (N - N_t^P) \left(\frac{\lambda_0}{N} + \sum_{k=1}^{N_t^P} \xi(k, \tau_k^P) \right) \quad (38)$$

In the zero-volatility intensity model, the intensity depends only on the default level. So the local intensity is the same as the original stochastic intensity.

2.7 The Child Default Intensity Process

The conditional probability that the next default is in child k given n total prior defaults in the parent and m_k prior defaults in child k is given by

$$\begin{aligned} C^k(n, m_k, \lambda_t, t) &= P(\tau_{n+1}^P = \tau_{m_k+1}^k | \tau_n^P < t, \bar{F}_t^P \cup \bar{F}_t^k) \\ &= \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) C_k(n, \bar{m}, t) / \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) \end{aligned} \quad (39)$$

where $\mathfrak{S}_k(n, i) = \{\bar{m} \mid |\bar{m}| = n; m_k = i; 0 \leq m_j \leq M_j, j = 1, \dots, K\}$ is the set of child default state subjected to the constraint that the parent default count is n and child k default count is $m_k = i$.

The next-to-default intensity of child k , conditional on m_k prior defaults is

$$\mu^k(m_k, t) = \sum_{n=m_k}^{N-M_k+m_k} \lambda(n, t) C^k(n, m_k, t) \quad (40)$$

It is clear that by construction, we have $\mu^k(m_k = M_k, t) = 0$, meaning that once all credits in child k have defaulted, its default intensity drops to zero.

The unconditional probability of n parent defaults and m_k defaults in the child k is

$$P^k(n, m_k, \lambda_t, t) = \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) \quad (41)$$

and the unconditional probability of m_k defaults in child k is

$$P^k(m_k, \lambda_t, t) = \sum_{n=m_k}^{N-M_k+m_k} P^k(n, m_k, \lambda_t, t) \quad (42)$$

2.8 Local Intensity Model and Stochastic Intensity Model

It has been reported that tranche pricing is the same under the LI model and the SI model [1], and that they result in different prices for forward starting tranches and tranche options [1][16]. In this section, we illustrate that the LI model and the SI model are equivalent for pricing static products such as tranches thanks to the Gyongy's theorem. But they are different for pricing dynamic products such as tranche option, forward starting and Leveraged-Super-Senior (LSS).

For simplicity and ease of notation, we use the single portfolio setting to explain the reason why the local intensity model is not equivalent to the stochastic intensity model. The multi-portfolio setting follows essentially the same line.

In Sec. 2.5, we showed that the LI process, which can be considered as the Markovian projection of the SI process [19] and has the same marginal default density as the SI process according to the Gyongy's theorem, is conditioned on today's information filtration \tilde{F}_0^P . The local intensity conditional on future filtration will be different. To emphasize the point, we express the local intensity conditioned on a filtration of future time s by

$$\begin{aligned}
G(N_t^P, t; N_s^P, \lambda_s) &= E\{\lambda_t | N_t^P, \tilde{F}_s^P\} = \\
\int_0^\infty \lambda P(\lambda | N_t^P, N_s^P, \lambda_s) d\lambda &= \int_0^\infty \lambda \frac{P(N_t^P, \lambda | N_s^P, \lambda_s)}{P(N_t^P | N_s^P, \lambda_s)} d\lambda
\end{aligned} \tag{43}$$

where $t > s$ and $\tilde{F}_s^P = \sigma\{\tau_k^P \leq s\} \cup \{\lambda_u | u \leq s\}$ is the time s filtration of the parent default and intensity and default. $P(N_t^P, \lambda_t | N_s^P, \lambda_s)$ is the SI model probability density at time t conditional on \tilde{F}_s^P .

In the zero-volatility intensity model, the default intensity depends on the current default level N_t^P only, and is independent of $N_s^P, \lambda_s; s < t$. As a result, we conclude that in the zero-volatility intensity model, the conditional local intensity and the (unconditional) local intensity are the same,

$$G^{Zero-Vol}(N_t^P, t; N_s^P, \lambda_s) = E\{\lambda_t | N_t^P, \tilde{F}_s^P\} = E\{\lambda_t | N_t^P\} = G^{Zero-Vol}(N_t^P, t) \tag{44}$$

Later in this section, we will show that Eq. (44) is the reason that the LI model and SI model are equivalent for the zero-volatility intensity model.

The LI model default level probability density conditional on the future time s is given by

$$P(N_t^P | N_s^P, \lambda_s) = \int_0^\infty P(N_t^P, \lambda | N_s^P, \lambda_s) d\lambda \tag{45}$$

By the law of iterative expectation, the local intensity seen today is equal to⁶

$$\begin{aligned}
G(n, t) &= G(N_t^P = n, t; N_0^P = 0, \lambda_0) = E\{G(N_t^P, t; N_s^P, \lambda_s) | N_t^P = n, \tilde{F}_0^P\} \\
&= \sum_{m=0}^n \int_0^\infty G(N_t^P = n, t; N_s^P = m, \lambda_s) P(N_s^P = m, \lambda_s | N_t^P = n) d\lambda_s
\end{aligned} \tag{46}$$

Eq. (43) shows that the time s local intensity is the expectation of the time t local intensity conditioned on the time s filtration \tilde{F}_s^P . Eqs. (43) and (46) show that (a) the local intensity seen from a future time is not equivalent to the local intensity seen today, and (b) the today's local intensity is the expected future local intensity. This suggests that the LI model is equivalent to the SI model for pricing static products which depend only on today's information, but they are not equivalent when pricing products with payoff depending on two or more future times⁷.

⁶ The today's local intensity is analogous to the Dupire's local volatility function for the equity model ([5]).

⁷ This is similar to equity derivative pricing where the LV and SV models are equivalent in terms of pricing vanilla European options, but differ for forward starting option. Piterbarg [19] pointed out that the

Suppose a payoff function $F(N_T^P, T)$ that depends on the default level at time T only. Assuming zero interest rate, the expected payoff is

$$E\{F(N_T^P, T) \tilde{F}_0^P\} = \sum_{n=0}^N \int_0^\infty F(n, T) P(n, \lambda_T) d\lambda_T = \sum_{n=0}^N F(n, T) P(n, T) \quad (47)$$

where $P(n, T)$ is the time $t = 0$ local intensity probability density defined in Eq. (27).

Hence, the expected payoff is the same under the LI model and SI model. Since the tranche price is just a weighted average of the expected payoff (47), we conclude that the LI and SI models are equivalent for tranche pricing.

Now, let's suppose that the payoff function $H(N_T^P, N_U^P)$ depends on the default level at two future times $T < U$. Under the SI model, the expected payoff is

$$\begin{aligned} V^{SI}(T, U) &= E\{H(N_T^P, N_U^P) \tilde{F}_0^P\} = \sum_{n=0}^N \sum_{m=0}^n \iint H(m, n) P(m, n, \lambda_T, \lambda_U) d\lambda_T d\lambda_U \\ &= \sum_{n=0}^N \sum_{m=0}^n H(m, n) P^{SI}(n, U, m, T) \end{aligned} \quad (48)$$

where (the superscript SI denotes that the conditional probability density is under the SI model to differentiate with the same conditional probability $P^{LI}(n | m)$ under the LI model. We use n for N_U^P , and m for N_T^P to simplify notation.)⁸

$$\begin{aligned} P^{SI}(n, U, m, T) &= \int_0^\infty P(n, U, m, \lambda_T, T) d\lambda_T = P^{SI}(n, U | m, T) P(m, T) \\ P^{SI}(n, U | m, T) &= \int_0^\infty P(m, \lambda_T) P(n, U | m, \lambda_T, T) d\lambda_T / P(m, T) \\ P(n, U | m, \lambda_T, T) &= \int_0^\infty P(n, \lambda_U | m, \lambda_T, T) d\lambda_U \end{aligned} \quad (49)$$

On the other hand, under the LI model, the expectation in Eq. (47) becomes

$$V^{LI}(T, U) = E\{H(N_T^P, N_U^P) \tilde{F}_0^P\} = \sum_{n=0}^N \sum_{m=0}^n H(m, n) P^{LI}(n, U, m, T) \quad (50)$$

Markovian projection model, such as the LV model, differs from the original SV model in pricing of instruments that require sampling multiple points in time.

⁸ We sometime omit U and T under the implicit assumption that U is associated with n and T with m.

Comparing expression (48) with (50), we see that for $V^{SI}(T, U) = V^{LI}(T, U)$ to hold, we must have $P^{LI}(n, U, m, T) = P^{SI}(n, U, m, T)$ for $\forall((m, T), (n, U))$. In the following, we will show that this is generally not the case.

Applying Eq. (29) to the time T filtration \tilde{F}_T^P , we see that $P(n, U | m, \lambda_T, T)$ is the solution to the “forward” local intensity model equation for time $U > T$,

$$\begin{aligned} \frac{dP^{SI}(n, U | m, \lambda_T)}{dU} &= -G(n, U; m, \lambda_T)P^{SI}(n, U | m, \lambda_T) \\ &+ G(n-1, U; m, \lambda_T)P^{SI}(n-1, U | m, \lambda_T) \end{aligned} \quad (51)$$

where $G(n, U; m, \lambda_T)$ is the local intensity conditional on \tilde{F}_T^P , defined in Eq. (43).

Multiplying Eq. (51) by $P(N_T^P, \lambda_T)$ and integrating the result over λ_T , we obtain the equation for the SI model joint default level probability density

$$\begin{aligned} \frac{dP^{SI}(n, U, m, T)}{dU} &= -P^{SI}(n, U, m, T) \int_0^\infty G(n, U; m, \lambda_T) \frac{P(n, U, m, \lambda_T)}{P^{SI}(n, U, m, T)} d\lambda_T \\ &+ P^{SI}(n-1, U, m, T) \int_0^\infty G(n-1, U; m, \lambda_T) \frac{P(n-1, U, m, \lambda_T)}{P^{SI}(n-1, U, m, T)} d\lambda_T \end{aligned} \quad (52)$$

Under the LI model, the conditional default probability density satisfies Eq. (29),

$$\frac{dP^{LI}(n, U | m, T)}{dU} = -G(n, U)P^{LI}(n, U | m, T) + G(n-1, U)P^{LI}(n-1, U | m, T) \quad (53)$$

Multiplying Eq. (53) by $P(m, T)$ and noticing that $P^{LI}(n, m) = P^{LI}(n | m)P(m)$, we obtain the equation governing the LI model joint default level probability density

$$\frac{dP^{LI}(n, U, m, T)}{dU} = -G(n, U)P^{LI}(n, U, m, T) + G(n-1, U)P^{LI}(n-1, U, m, T) \quad (54)$$

Except for the zero-volatility intensity model, we obtain⁹

$$G(n, U) \neq \int_0^\infty G(n, U; m, \lambda_T) \frac{P(n, U, m, \lambda_T)}{P^{SI}(n, U, m, T)} d\lambda_T \quad (55)$$

This means that except for the zero-volatility intensity model, we have

⁹ Under the zero-volatility intensity model, Eq. (55) becomes Eq. (44).

$$P^{SI}(n, U, m, T) \neq P^{LI}(n, U, m, T) \quad (56)$$

which leads to

$$V^{LI}(T, U) \neq V^{SI}(T, U) \quad (57)$$

Therefore, the pricing of products whose payoffs depend on the default level at two or more future times is different under the LI model from that under the SI model if the SI model has a non-zero diffusion term (or equivalently, the probability density depends on λ_t in a non-trivial way. It will be shown in the next section that the forward starting tranche and tranche option fall into this category and hence have different pricing under the LI model and SI model, as has been confirmed by the numerical results presented in [1] and [16].

Remark 2.8.1: It is helpful to understanding the difference between the LI and SI models for pricing dynamic products using equity derivative analogy. The LI model is analogous to the local volatility (LV) model in equity derivative pricing. Hagan *et al* (2002) showed that the LV model, while having the same marginal distribution as SV model, predicts the implied volatility dynamics that is opposite of the behavior observed in the market. This suggests that LV model is not suitable for pricing products like the forward starting option for which the volatility dynamics is important. The implied volatility dynamics of the LV model is different from that of the SV model. However, both LV and SV model are calibrated to today's implied volatility surface. From the mathematical standpoint, the vanilla options are similar to tranches and the forward starting options are similar to the forward starting tranches. This is not to say that the credit and equity derivatives are the same. They are not. For one thing, the equity market exhibits a negative implied volatility skew where the implied volatility decreases as the strike increases. The tranche option has positive implied volatility skew where the equivalent Black implied volatility increases with increasing strike.

3 Product Pricing

In this section, we outline the steps for pricing bespoke tranche, forward starting tranche and tranche option. The CDS tranches are sometimes called static products in the sense of straight bonds or vanilla European options. Static credit products may be loosely defined as those instruments whose pricing is not affected by spread volatility. Products which cannot be classified as static then are dynamic products. Under this definition, spread volatility must have non-negligible impact on the pricing of dynamic products. Having given the above loose definition, we note that static models have been used to price dynamic credit products such as forward starting tranche [19].¹⁰ In the following, we use portfolio as the generic term for a bespoke portfolio which can be either the parent or a child in our multi-portfolio framework.

¹⁰ In equity derivative modeling, the LV model is popular among practitioners for exotic options such as cliquet options, ladder options even though the LV model is known to under-price.

3.1 CDS Tranche

A CDS tranche is a contract between two parties where one party sells credit protection and the other buys protection on a specific portion in the capital structure of a pool of synthetic CDSs. The tranche is defined by the attachment point (AP) and the detachment point (DP), maturity and the contract spread. The CDSs in the reference portfolio are referred to as credit, reference obligation or obligor. The protection seller receives a periodic payment from the protection buyer based on the tranche's remaining notional amount and the tranche spread. In exchange for the coupon payments, the protection seller is obligated to make contingent payment to the protection buyer for the portfolio cumulative default loss exceeding the AP. The total contingent payment is capped at $DP - AP$.

Let $L_t^{PF} = (1 - R)N_t^{PF} / N^{PF}$ be the cumulative percentage loss on the reference bespoke portfolio (PF), the tranche loss function at time t is defined as

$$L_t^{TR} = f(L_t^{PF}) = (L_t^{PF} - AP)^+ - (L_t^{PF} - DP)^+ \quad (58)$$

where the superscript PF denotes portfolio. Note that the portfolio itself is considered as the 0-100% tranche in which case we have $L_t^{TR} = L_t^{PF}$.

The tranche loss function (58) suggests that CDO tranche can be viewed as a long call option on the cumulative portfolio loss with the strike AP and a short call option with the strike DP. Using the equity derivative terminology, the tranche can be viewed as a bull call spread on the cumulative portfolio loss.

Assuming independence between the interest rate and the default level, and neglecting accrued premium, the mark-to-market value (MTM) to the tranche protection seller is

$$\begin{aligned} MTM_{Tranche} = & P_{Upfront} + S \sum_{k=1}^K D(T_k) \{ DP - AP - E_t [R(T_k) + L_{T_k}^{TR}] \} \Delta T_k \\ & - \sum_{k=1}^K \frac{1}{2} [D(T_{k-1}) + D(T_k)] E_t (L_{T_k}^{TR} - L_{T_{k-1}}^{TR}) \end{aligned} \quad (59)$$

In Eq. (59), $R(t) = [RN_t^{PF} / N^{PF} - (1 - DP)]^+$ is the amortizing amount removed from (the top of) the portfolio. Clearly, $R(t)$ can be non-zero only for tranches with $DP > 1 - R$. So it affects only the most senior tranche. For the whole portfolio where $AP = 0$ and $DP = 100\%$, we have $R(T) + L_T^{TR} = N_T^{PF} / N^{PF}$, the portfolio notional loss due to default. In Eq. (59), $P_{Upfront}$ is the upfront price quote and S is the tranche running spread. The market convention for investment grade (IG) is to quote equity tranche by an upfront price plus a fixed running spread (500 bps), and to quote mezzanine and senior tranches in all running spread form. Emerging market (EM) and High Yield (HY) indices quote price in upfront

with no running spread. Compared with the all running spread tranche, upfront payment reduces the spread sensitivity (DV01) and P&L convexity.

Eqs. (58) and (59) show that tranche pricing essential involves the calculation of

$$\begin{aligned} E(L_T^{PF} - A)^+ &= E\left[(1-R)N_T^{PF} / N^{PF} - A\right]^+ \\ &= \sum_{\bar{N}_T^C} \left[(1-R)N_T^{PF} / N^{PF} - A\right]^+ P(N_T^P, \bar{N}_T^C, T) \end{aligned} \quad (60)$$

with

$$P(N_T^P, \bar{N}_T^C, T) = \int_0^\infty P(N_T^P, \bar{N}_T^C, \lambda_T, T) d\lambda_T.$$

The multi-portfolio default level density $P(N_T^P, \bar{N}_T^C, T)$ can be obtained by solving Eq. (26). Eq. (60) shows that tranche pricing essentially requires only the marginal default level distribution as demonstrated numerically in [1]. In other words, the diffusion component in the stochastic intensity model does not matter. Zero-volatility model is sufficient. Since the diffusion term is irrelevant in tranche pricing, the diffusion volatility parameters in the SI model cannot be determined from calibrating to the tranche prices. These observations have been pointed out in [1] and [16].

The fact that tranche pricing does not require intensity volatility suggests that it is more efficient to fit tranche prices using the distribution density $P(N_t^P, \bar{N}_t^C, t)$ for the Markovian projection model rather than the original full blown stochastic intensity model.

3.2 Forward Starting Tranche

A forward starting tranche is a forward contract on a tranche that starts at a future date with predetermined notional amount, maturity, reference portfolio, attachment and detachment points, and tranche premium. At the tranche starting date, the AP and DP are adjusted according to the portfolio loss such that the tranche subordination remains unchanged. Suppose that the AP and DP for the forward starting tranche are a and b , and the forward starting date is T_0 . If the cumulative portfolio loss at T_0 is $L_{T_0}^{PF}$, the AP and DP of the forward starting tranche at the starting date T_0 are $A = a + L_{T_0}^{PF}$ and $B = b + L_{T_0}^{PF}$. The effective tranche subordination is still a and the effective tranche size remains to be $b - a$. Both A and B are capped at 100%. After T_0 , it is a regular tranche.

Let the forward starting tranche maturity be T_N , The MTM value before the starting date $t < T_0$ to the protection seller is equal to the present value of the difference over the life

of the tranche between the expected tranche loss and the expected premium payment. Assuming unit notional amount and option contract spread S , this is given by¹¹

$$MTM(t) = S \sum_{k=1}^N E_t \left(L_{T_k}^{TR} - L_{T_{k-1}}^{TR} \right) D_{T_k} - \sum_{k=1}^N E_t \left(b - a - L_{T_{k-1}}^{TR} \right) D_{T_k} \quad (61)$$

where the tranche loss at $t > T$ defined as

$$L_t^{TR} = \left(L_t^{PF} - a - L_T^{PF} \right)^+ - \left(L_t^{PF} - b - L_T^{PF} \right)^+ \quad (62)$$

Eq. (62) shows that the the payoff of a forward starting tranche is similar to that of a forward starting equity call spread with portfolio loss L_t^{PF} serving the role of stock price. Valuation of the forward starting tranche reduces to calculating the expectation

$$\begin{aligned} E \left(L_t^{PF} - A - L_T^{PF} \right)^+ &= E \left[(1 - R) \left(N_t^{PF} - N_T^{PF} \right) / N_T^{PF} - A \right]^+ \\ &= \sum_{\bar{N}_t^C, \bar{N}_T^C} \int P \left(N_t^P, \bar{N}_t^C \mid N_T^P, \bar{N}_T^C, \lambda_T \right) P \left(N_T^P, \bar{N}_T^C, \lambda_T \right) \left(L_t^{PF} - A - L_T^{PF} \right)^+ d\lambda_T \end{aligned} \quad (63)$$

where

$$P \left(N_t^P, \bar{N}_t^C \mid N_T^P, \bar{N}_T^C, \lambda_T \right) = \int P \left(N_t^P, \bar{N}_t^C, \lambda_t \mid N_T^P, \bar{N}_T^C, \lambda_T \right) d\lambda_t \quad (64)$$

The exposition in Sec. 2.8 indicates that the expectation in (63) is different under the SI model from that under the LI model. Therefore, the SI model and LI model are not equivalent for forward starting tranches.

Remark 3.2.1: Equity forward starting options have been priced using, in decreasing level of sophistication, SV model, LV model, and BS model with a suitably chosen implied volatility. The SI model with non-zero volatility $\sigma(\lambda_t, N_t^P, t)$ is compared to the SV model where the default intensity plays the role of volatility. The LI model is the counterpart to the LV model. Factor copula models with constant correlation correspond to the BS model with an implied volatility [14].

3.3 CDS Tranche Option

Option on tranche is a European option that gives the option holder the right but not obligation to buy or sell credit protection at a future time on a predetermined CDO tranche with a specified spread (strike). Unlike the single name CDS option which ceases to exist after default, the tranche option does not knock out after a default in the portfolio. Upon exercising the option, the option buyer can settle for tranche loss occurred prior to the option expiry. The premium payment is based on the remaining tranche notional amount.

¹¹ For simplicity, we assume all payments are made at the payment date.

Let the option expiry be T_0 , the underlying tranche maturity be T_N , and the option strike be S . The value of a payer tranche option, which gives the holder the right to buy credit protection and to pay the spread S , is equal to¹²

$$D_{T_0} E_0 \left\{ \sum_{j=1}^N \frac{D_{T_j}}{D_{T_0}} \left[E_{T_0} (L_{T_j}^{TR}) - E_{T_0} (L_{T_{j-1}}^{TR}) \right] - S \sum_{j=1}^N \frac{D_{T_j}}{D_{T_0}} \left[AD - E_{T_0} (L_{T_j}^{TR}) \right] \Delta T_j + L_{T_0}^{TR} \right\}^+ \quad (65)$$

where the tranche loss function L_t^{TR} is defined by Eq. (58), and $AD = DP - AP$. $L_{T_0}^{TR}$ is the tranche loss before the option expiry, which the option buyer will settle upon exercising the option.

It is seen from Eq. (60) that the tranche option price depends on the joint default distribution at option expiry T_0 and coupon payment time T_j . Therefore, a stochastic intensity model will yield a different price than its corresponding local volatility model. Examples of numerical difference in pricing of tranche options between SI model and LI model can be found in references [1] and [16].

4 Calibration

The model calibration is to fit the parent default intensity and the child conditional default probability function $g_k(x_k)$ to the liquid market prices of tranches which reference either the parent or the children. We consider two fundamental situations:

- 1) Portfolio Thinning in which the bespoke portfolio is a child of an index.
- 2) Portfolio Enlargement where the bespoke portfolio is the parent that includes one or more indices as children.

4.1 Portfolio Thinning

When the parent is an index which contains the bespoke, we fit the parent default intensity $\lambda(N_i^P, t)$ to standard tranche prices independent of the children, plus liquid tranche option prices if available. The volatility and mean-reversion parameter calibration requires option prices. This is similar to the interest rate modeling where the term structure of bond price (or yield curve) cannot determine the volatility and mean-reversion rate. Absence of tranche option or forward starting tranche prices, one is left with the choice either using a zero-volatility model which fits the tranche well or make assumptions on the default intensity volatility and mean-reversion rate. The actual calibration procedure is model dependent.

¹² To simplify notion, we assume that the tranche loss in ΔT_i is paid out at T_i and no accrued coupon.

Having determined $\lambda(n, t)$, we next calibrate the child conditional default probability $C_k(N_t^P, \bar{N}_t^C, t)$ for all children. As stated earlier, they must be calibrated by simultaneously solving for $\xi_k(t), k = 1, \dots, K$. The children are bespoke portfolios and unlikely to have liquid tranches referencing them. Therefore, we fit the model spread for each child to the intrinsic spread of that child. The child intrinsic spread for a maturity can be calculated from the constituent CDS curves in the child. Since we have calibrated the parent intensity to fit the parent intrinsic spread, there are only $K-1$ independent child spreads available for child calibration. So we have $K-1$ equation for K ξ 's. The extra equation is provided by the restriction that child conditional default probability must sum to 1 as they must satisfy the constraint Eq. (10).

4.2 Portfolio Enlargement

The case in which the bespoke portfolio is the parent is potentially more important. From the practical standpoint, a bespoke reference portfolio often has only a portion of credits overlapping with a particular index. In this case, the first step is to construct a (parent) portfolio that is the union of the index and the bespoke. The index is considered a child and the credits in the bespoke but not in the index forms another child. Now, our problem consists of a bespoke parent that contains indices as children.

When the bespoke portfolio is the parent of index children, the parent default intensity λ_t cannot be directly fitted to the index tranches because the parent contains more credits than do the tranche reference portfolios. In this case, we fit the multi-portfolio default model to the tranches referencing (child) index portfolio. to this end, we must calibrate simultaneously the parent default intensity λ_t and the child conditional default probability $C_k(N_t^P, \bar{N}_t^C, t), k = 1, \dots, K$.

Specifically, we consider the case in which there is only one child is an index. Multiple index children case can be dealt with in a similar fashion and will be briefly discussed in the next section. This is the two-child case where the index is a child and the rest of the bespoke portfolio forms the other child. The calibration procedures are as follows:

- [1] Specify the parent default intensity model. Any top down intensity model can be a candidate. At this stage, the intensity λ_t is yet to be calibrated.
- [2] Specify the child portfolio conditional default probability $C_k(n, \bar{m}, t)$ in terms of Eqs. (12), (13) and (14) which satisfy the constraints Eqs. (8), (10) and (11).
- [3] Fit λ_t and $\xi_k(t), k = 1, \dots, K$ to the liquid tranches referencing index child.

Note that calibration of volatility parameters (mean reversion and diffusion volatility) requires spread volatility products such as tranche options. Absence of such volatility product, we can assume values for these parameters or simply use the zero-volatility intensity model which can be solved efficiently using the method described in Appendix.

5 Model Applications and Limitations

5.1 Bespoke Tranche Referencing Two Indexes

As stated earlier in this paper, the goal of our model is to establish a relation between the bespoke portfolio and some relevant indexes so we can incorporate the index pricing information into our bespoke tranche pricing. In other words, we want to value the bespoke tranche in a way consistent with the pricing of some standard tranches and indexes. In this section, we discuss model application through a practical example.

Example: A 3-7% STCDO referencing 100 obligors of which 50 belong to the iTraxx index and the other 50 are members of CDX.NA.IG index.

The standard method to price such a bespoke STCDO is the one-factor copula model with base correlation (BC) mapping. For this case, the indices to which the bespoke is mapped are naturally the CDX index and iTraxx index. Base correlation mapping can be done in several ways described in [2] [21]. For example, the BC for the bespoke equity tranche is the weighted average of the BCs of the index equity tranches with the same maturity and detachment point, where the weights for CDX and iTraxx are 0.5. Since 7% is not a standard iTraxx tranche detachment point, the BC for 7% DP in the iTraxx must be interpolated from the BCs for the standard DPs. The interpolation of DP should ensure that the equity expected loss be an increasing function of the detachment point, which may be difficult.

We take an alternative approach to building a relation between the bespoke reference portfolio and the indices. First we enlarge the portfolio such that the enlarged portfolio (parent) contains both CDX and iTraxx indexes as children. We then shrink the parent to the bespoke portfolio which contains only a portion of the indices. The procedure is as follows:

- 1) Assume a top-down default process for the “global” portfolio that is the union of CDX and iTraxx indices. At this stage, the model is still undetermined in the sense that the model parameters remain undetermined.
- 2) Assume for each index, which is a child, a conditional default probability. Because the standard tranches referencing the two indices are generally different, the conditional default probability is distinct for each index.
- 3) Jointly calibrate the default intensity for the “global” portfolio and the conditional default probabilities for the two children to the standard tranches referencing CDX and iTraxx using the procedure described in section 4.2.
- 4) Having calibrated the default intensity process for the “global” portfolio, we repartition the parent into the bespoke reference portfolio and the second child, the complementary portfolio, which contains the names that are in the “global” portfolio but not in the bespoke. In other word, we thin the parent process to that of bespoke.
- 5) Calculate the bespoke default probability and tranches referencing the bespoke portfolio.

Steps 1 – 3 above can be considered as portfolio enlargement process that creates a default process for the universal portfolio. This universal portfolio contains both the bespoke portfolio and additional names in order to incorporate the standard tranche prices into the bespoke tranche pricing. Step 4 is a thinning process from the universal portfolio to the bespoke.

At first glance, the above way to obtain the bespoke portfolio default process may seem convoluted. In our view, this is a sensible way to link the bespoke portfolio to the CDX and iTraxx indices which have substantial overlap – 40% in this case – with our bespoke portfolio where both CDX and iTraxx have 125 names. So 40% of the indexes are in the bespoke portfolio. The most important is that through this way, we obtain a dynamic default process for the bespoke portfolio which cannot be achieved by base correlation mapping. The dynamic process can then be used to consistently price and to risk manage the bespoke tranche.

5.2 Credit Substitution

Another potential application is the credit substitution. Managed bespoke tranche permits credit substitution where one or more credits in the portfolio are replaced with other credits of equal notional amounts. The purposes of the credit substitution can be one or a combination of the followings:

- 1) To improve or maintain the rating of the tranche.
- 2) To prevent credit deterioration.
- 3) To lock in gains on credits whose spreads have tightened substantially.

In practice, the effect of credit substitution on the tranche often involves an adjustment to the tranche subordination to account for the trading gain or loss caused by the credit substitution without adjusting the tranche spread. The idea is that the expected loss on the tranche must remain the same before and after the credit substitution. When a credit with a higher spread is replaced by another credit with a lower spread, the bespoke portfolio becomes less risky and the tranche subordination must be adjusted downwards to reflect the reduced portfolio risk. Conversely, if a lower spread credit is replaced with a higher spread credit, the subordination must be adjusted upwards to account for the increased portfolio risk.

Under the factor copula model framework, calculation of the subordination adjustment entails to first calculate the base correlations for the new tranche using a base correlation mapping scheme, and then calculate the expected loss of the new tranche. Adjust the new tranche subordination until the post substitution tranche expected loss is equal to that before substitution or the difference is minimized within some predetermined tolerance.

Halperin and Tomecek [11] suggested an interesting method for the credit substitution. First, the single name credit default dynamic process consistent with the portfolio default dynamics is obtained by random thinning. Second, the default probabilities of the single names of the new portfolio are inferred, and the top-down default dynamic process of the

new portfolio is updated. The expected loss of the tranche after credit substitution can be calculated from the new portfolio default process. See their paper for more details.

Assume that the pre-substitution portfolio consists of credit sets A and B where the credits in set B are replaced with credits in set C . In our model framework, the expected loss on the new tranche referencing portfolio $A \cup C$ is evaluated as follows:

- 1) Construct the dynamic default processes for the overall portfolio $A \cup B \cup C$, and sub-portfolio $A \cup B$ and $A \cup C$ using the method described above.
- 2) The pre-substitution tranche expected loss is calculated from the portfolio $A \cup B$.
- 3) The post substitution tranche expected loss is calculated from the portfolio $A \cup C$.
- 4) Adjust the subordination of the post substitution tranche until the post substitution tranche expected loss is equal to that of the pre-substitution, or the error is minimized.

6 Conclusions

We have presented a multi-portfolio model for the pricing of bespoke portfolio CDO tranche. A fundamental model assumption is that the bespoke portfolio has a child-parent relationship with an index or multiple indices. The bespoke portfolio may be the child of an index, or it may be the parent of one or several indices. This assumption enables to establish a relationship between the bespoke portfolio default process and the index default process. As a result, the bespoke tranche pricing is consistent with the index tranche pricing.

We have taken as given the default model for the parent portfolio. There are many choices for such a model in the open literatures ([1], [4], [6], [12], [16]). We have proposed a model for distribution of the parent default among the children. This model is the simplest while satisfying all the required constraints. The default process in the children is a combination of the default process in the parent and the conditional probability of default in the children.

The model is calibrated to the index tranches and the available instruments of the bespoke portfolio. The calibration is considerably simpler when the parent portfolio is an index than when the bespoke portfolio is the parent. In the former case, bespoke default process can be built in two separate steps. First, the parent default process is calibrated to the index without regarding the child portfolio. Second, the conditional probability is calibrated to the available instruments of the bespoke portfolio. In the latter case where the bespoke portfolio is the parent and the index is the child, we must build the default process for the bespoke portfolio and calibrate it to the index tranches as well as the available prices of the bespoke portfolio. In this case, the parent default process is likely the one that is of interest. Once calibrated, the model can be used to price any bespoke instruments.

Finally, we note that this paper is the methodology part of the model. It does not address the numerical implementation although major components of implementation have been outlined here and in the references cited. Numerical results will be presented in a separate

paper which will use both PDE and Monte Carlo simulation depending on the choice of the parent default model.

References

- [1] M. Arnsdorf and I. Halperin (2007), “BSLP: Markovian Bivariate Spread-Loss Model for Portfolio Credit Derivatives”, JP Morgan, 2007.
- [2] P. Baheti and S. Morgan (2007), “Base Correlation Mapping”, Lehman Brothers, March 7, 2007.
- [3] T. Bielecki, S. Crepey and M. Jeanblanc (2008), “Up and down credit risk”, www.defaultrisk.com.
- [4] X. Ding, K. Giesecke and P. Tomecek (2008), “Time-Changed Birth Processes and Multi-Name Credit Derivatives”, Stanford University.
- [5] B. Dupire (1994), “Pricing With A Smile”, Risk January 1994, p18-20.
- [6] E. Errais, K. Giesecke and L. Goldberg (2009), “Affine Point Processes and Portfolio Credit Risk”, Working paper, www.defaultrisk.com.
- [7] K. Giesecke (2007), “The correlation-neutral measure for portfolio credit”, Working paper.
- [8] K. Giesecke and B. Kim (2007), “Estimating Tranche Spreads By Loss Process Simulation”, Proc. 2007 Winter Simulation Conference, Henderson *et al* eds.
- [9] K. Giesecke, L. Goldberg and X. Ding (2009), “A Top Down Approach to Multi-Name Credit”, Working Paper, April 2009.
- [10] P. Hagan, D. Kumar, A. Lesniewski, and D. Woodward (2002), “Managing smile risk”, WILMOTT Magazine, September, pp. 84–108.
- [11] I. Halperin and P. Tomecek (2008), “Climbing Down from The Top: Single Name Dynamics in Credit Top Down Models”, JP Morgan, August 21, 2008.
- [12] A. Herbertsson (2007), “Pricing Synthetic CDO Tranches in A Model with Default Contagion Using The Matrix-Analytic Approach”, www.defaultrisk.com.
- [13] L. Ingber (2008), “Adaptive simulated annealing (ASA)”.
- [14] K. Jackson and W. Zhang (2007), “Valuation of Forward Starting CDOs”, Working paper, University of Toronto.

- [15] A. Lopatin (2008), “A simple dynamic model for pricing and hedging heterogeneous CDOs”, November 1, 2008, www.defaultrisk.com.
- [16] A. Lopatin and T. Misirpashaev (2007), “Two-dimensional Markovian model for dynamics of aggregate credit loss”, May 3, 2007, www.defaultrisk.com.
- [17] C. Moler and C. Loan (2003), “Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years later”, *SIAM Review*, Vol. 45, No. 1, pp. 3-100.
- [18] S. Morgan and A. Mortensen (2007), “CDO Hedging Anomalies in the Base Correlation Approach”, Lehman Brothers, October 29, 2007.
- [19] V. Piterbarg (2007), “Markovian Projection Method for Volatility Calibration”, Risk Magazine, April (2007).
- [20] L. Schloegl, A. Mortensen and S. Morgan (2008), “Strange Risk in a Correlation Model”, Lehman Brothers, March 31, 2008.
- [21] J. Turc, D. Benharmou, B. Herzog and M. Teyssier (2006), “Pricing Bespoke CDOs: Latest Development”, Societe Generale Credit Research, July 28, 2006.
- [22] S. Amraoui and S. Hitier (2008), “Optimal Stochastic Recovery for Base Correlation”, BNP Paribas, www.defaultrisk.com.

Appendix: The Two-Child Case

One important practical case is where there are two sub-portfolios. The bespoke portfolio can be either the parent or one of the children. In the appendix, we describe an algorithm for efficient solution of the local intensity model equation (26) for the two-child case.

When $K = 2$, a default must be in either child 1 or child 2. We have $n = u + v$ and $N = M_1 + M_2$ where u is the default level in child 1 and v is the default level in child 2. To lighten notation, we omit the dependence on n and suppress t . So $P(n, \bar{m}, t) = P(u, v)$ and $C_k(n, \bar{m}, t) = C_k(u, v)$ where $k = 1, 2$. The local intensity $G(n, t)$ in Eq. (27) is a function of the parameters of the stochastic intensity λ_t .

Using the notation defined above, Eq. (26) becomes

$$\frac{dP(u, v)}{dt} = -G(n)P(u, v) + G(n-1)\{C_1(u-1, v)P(u-1, v) + C_2(u, v-1)P(u, v-1)\} \quad (\text{A.1})$$

Define $W = (M_1 + 1)(M_2 + 1)$, the total number of $P(u, v, t)$, and Q a W -dimensional vector where the component of vector Q is defined by the mapping

$$Q_{J(u, v)}(t) = P(u, v, t), \quad \text{with} \quad J(u, v) = u(M_2 + 1) + v \quad (\text{A.2})$$

subjected to the constraints

$$0 \leq u \leq M_1 \quad \text{and} \quad 0 \leq v \leq M_2 \quad (\text{A.3})$$

Using mapping (A.2), Eq. (A.1) can be rewritten as

$$\frac{dQ_j(t)}{dt} = a_{jj}(t)Q_j(t) + a_{j-1, j}(t)Q_{j-1}(t) + a_{j-(M_2+1), j}(t)Q_{j-(M_2+1)}(t) \quad (\text{A.4})$$

where $j = J(u, v)$, $j-1 = J(u, v-1)$, $j-(M_2+1) = J(u-1, v)$, and the coefficient matrix $A = (a_{ij})_{W \times W}$ is defined by

$$\begin{aligned} a_{jj}(t) &= -G(n, t), \\ a_{j-(M_2+1), j}(t) &= G(n-1, t)C_1(u-1, v, t), \\ a_{j-1, j}(t) &= G(n-1, t)C_2(u, v-1, t), \\ a_{ij}(t) &= 0, \quad \text{Otherwise} \quad . \end{aligned} \quad (\text{A.5})$$

The initial condition for Eq. (A.4) is

$$Q_j(0) = \delta_{j0}. \quad (\text{A.6})$$

Note that row $j [= J(u, v)]$ of matrix A contains three non-zero entries, a_{jj} , $a_{j, j+1}$ and $a_{j, J(u+1, v)}$ where $J(u, v+1) = j+1$ and $J(u+1, v) = j + M_2 + 1$. It is easy to verified that coefficient matrix A satisfies the property

$$a_{jj} = -\sum_{i \neq j}^W a_{ji} = -a_{j, j+1} - a_{j, j+(M_2+1)}, \quad j = 0, \dots, W-1 \quad (\text{A.7})$$

and hence is a generator matrix.

Eq. (A.4) can be rewritten conveniently in the matrix form

$$\frac{dQ}{dt} = QA \quad (\text{A.8})$$

Since the generator matrix A is deterministic, the solution of equation (A.8) is

$$Q(t) = Q(t_0) \times \text{Exp} \left(\int_{t_0}^t A(s) ds \right) \quad (\text{A.9})$$

The matrix exponential in formula (A.9) can be computed using the Pade approximation. Moler and Loan (2003) provided an overview of various methods for computing the matrix exponential.