

## A Multi-Portfolio Model for Bespoke CDO Pricing Part I: Methodology

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## A Multi-Portfolio Model for Bespoke CDO Pricing

### Part I: Methodology

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**Abstract:** We propose a dynamic multi-portfolio default model for pricing synthetic CDO tranches referencing a bespoke portfolio. The key model assumption is the child-parent relationship between the bespoke and the overall parent portfolios. The defaults in the children are driven by the default in the parent. The parent default can be modeled using any suitable top-down dynamic model. The parent defaults are distributed amongst the children according to the conditional child default probability. The distribution of default state is determined by the parent default state and the conditional child default probability. Given the default state, bespoke tranche loss can be calculated.

**Keywords:** Dynamic Multi-Portfolio Default Model, Bespoke CDO Pricing.

**JEL Codes:** C5; G12; G13

## **1. Introduction**

Recent years have seen significant progress in the modeling of CDO tranches with the emergence of the top-down Markovian contagion, or self-affecting, dynamic credit default models. Top down refers to the modeling approach where the credits in the portfolio are assumed to have the same characteristics – notional amount, default probability, recovery rate – and hence only the number and timing of credit default are important to the calculation of the portfolio credit loss. The identity of the defaulter is irrelevant. In mathematical term, at a given time, the natural filtration at time  $t$  generated by the portfolio default history contains the information about number and timing of default up to  $t$ , but not the identities of the defaulters. Credit default contagion refers to the model feature that the portfolio default intensity jumps post default, reflecting the empirical observation that the surviving credits in the portfolio generally become more likely to default after a default.<sup>1</sup> In other words, credit default has contagion effect on the financial health of the firms still alive.

In the reduced-form credit default contagion models, the default counting process is modeled by a self-affecting intensity process that accounts for the empirical evidence that defaults are clustered where more credit defaults are likely to follow after a default. The fact that the default contagion models, whether they are default only in which case spread volatility is due entirely to defaults [Herbertsson, 2008] or they account for stochastic spread volatility [Arnsdorf and Halperin, 2008; Ding et al. 2008; Giesecke and Goldberg, 2005; Lopatin and Misirpashaev, 2007], can achieve almost perfect calibration to the standard index tranches suggests that the default contagion is a major risk factor of the credit portfolio products, and any sensible portfolio default model must account for this effect in some way.

In addition to the near perfect calibration to the standard tranches, tranches referencing the index portfolio but with non-standard strikes and maturities can be priced and risk managed consistently using the same model. As a result, there is no longer the

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<sup>1</sup> Strictly speaking, surviving firms do not necessarily all become riskier after a default. For example, a firm may be stronger after a default by its competitor. But the systemic risk generally increases after default.

need to use the *ad hoc* base correlation interpolation/extrapolation schemes, thereby eliminating a major model inconsistency that is inherent in the current market standard one-factor copula pricing model [Morgan and Mortensen, 2007; Schloegl et al., 2008].

However, pricing and risk managing of tranches referencing bespoke portfolio still pose a tremendous challenge due to the illiquid nature of the bespoke portfolio. Complete model calibration to bespoke portfolio is difficult, if not impossible, because of lack of prices on tranches referencing the bespoke portfolio. The current market practice for bespoke tranche pricing is the one-factor Gaussian copula model with some form of base correlation mapping scheme between the base correlation of the standard tranche and that of the bespoke tranche [Baheti and Morgan, 2007; Turc et al., 2006]. Although these mapping schemes have some theoretical justifications, they are *ad hoc* in nature and known to admit arbitrage (Morgan and Mortensen, 2007; Schloegl et al., 2008).

Furthermore, the base correlation mapping is static which makes risk management of bespoke tranche problematic because different tranches of the same portfolio will have to be valued and hedged using different mapping. This problem is akin to hedging an equity option portfolio using different implied volatility. It is thus desirable to develop dynamic and consistent models that incorporate available index pricing information into pricing of bespoke tranche products.

In this paper, we present a dynamic multi-portfolio default model for the pricing of tranche referencing a bespoke portfolio. The purpose is to make use of the pricing information on indexes<sup>2</sup> that overlap with the bespoke portfolio such that the bespoke tranche is priced consistently with the indexes. To this end, we make a fundamental assumption that the bespoke portfolio has the child-parent relationship with an index or several indexes. A parent is defined as the overall portfolio that contains all credits. A child is a sub-portfolio consisting of a portion of the credits in the parent. As such, the bespoke portfolio must be a subset of the parent. The bespoke portfolio can be either the parent, or a child or the union of several children. We refer the former as (random)

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<sup>2</sup> Index is referred to as a reference portfolio of liquid tranches. Examples of index are CDX and iTraxx.

portfolio enlargement in the sense that the bespoke portfolio is an expansion of credit indexes, and the latter as (random) portfolio thinning where the bespoke portfolio is a contraction of an index. The portfolio thinning is similar to the random thinning to single name CDS proposed by Giesecke and Goldberg (2005), and includes the random thinning to single names as a special case when the children contain only one credit. In the portfolio enlargement, one of the children must be an index portfolio. In the portfolio thinning, the parent must be an index. More complex cases where the bespoke portfolio partially overlaps with the index can be handled using a combination of portfolio enlargement and thinning.

Through the child-parent relationship between the bespoke portfolio and the index, the standard tranche price information is incorporated into the bespoke portfolio default model and hence into the bespoke tranche pricing. This is another way of achieving the same goal as that of base correlation mapping between the standard tranche and bespoke tranche. While the motivation behind our model is the same as that of base correlation mapping, there are fundamental differences. First, the base correlation mapping is static in the sense that it applies only to each individual tranche, namely, the base correlation mapping is maturity, strike and reference portfolio specific. Different tranches need different mapping, creating inconsistency which is especially problematic for risk management. This consistency problem is similar to risk managing equity option portfolio using Black-Scholes model and Black implied volatility, where options of different expiries and strikes are valued using different implied volatilities making portfolio hedging problematic. Second, the base correlation mapping is not arbitrage free. Our model is dynamic in the sense that once the model is calibrated to the standard tranches, bespoke tranches are priced in a consistent and arbitrage-free manner, across all maturities and strikes.

The rest of the article is organized as follows. In section 2, we first define some symbols that will be used throughout this paper. We then describe the default processes for the parent and the child portfolios. We assume that the parent default process is given, and focus on the child portfolio conditional default probability distribution using the fact that child portfolio default process is derived from the parent default process and the

conditional default probability of child portfolio. Section 3 outlines the tranche and index pricing formulas that will be used in fitting the model to the market prices of the index and standard tranches. Section 4 gives the calibration procedures. If the parent portfolio is the index, we calculate the parent default process intensity independently of the child process by fitting to the index and standard tranches. We then calculate the child portfolio conditional default probability by fitting the child default process, which contains only the conditional default probability as unknown, to the child portfolio spread and other available liquid prices. If the child portfolio is the index and the parent portfolio is the bespoke, it is necessary to jointly calibrate the parent default process and the child portfolio conditional default probability. Section 5 discusses potential applications of the model, and section 6 concludes the paper.

## **2. Model Description**

### **2.1. Nomenclature**

The following notations are used throughout this paper.

- $\tau_1^P < \tau_2^P < \dots < \tau_N^P$ : Ordered default times in the parent which contains  $N$  obligors.
- $\tau_1^k < \tau_2^k < \dots < \tau_{M_k}^k$ : Ordered default times in child  $k$  which contains  $M_k$  obligors.
- $T_0 < T_1 < T_2 < \dots < T_k$ : Coupon payment date set.
- $\lambda(n, t) = \lambda(N_t^P = n, t)$ : The next-to-default (NTD) intensity of the parent.
- $\mathcal{F}_t^P = \sigma\{\tau_s^P \leq t\} \cup \{\lambda_s | s \leq t\}$ : The natural filtration generated up to time  $t$  by the default in the parent portfolio and the parent default intensity.
- $\mathcal{F}_t^k = \sigma\{\tau_j^k \leq t\}$ : Natural filtration generated by the default in the child  $k$ .
- $N_t^P = \text{Sup}\{k | \tau_k^P \leq t\}$ : The default level in the parent.
- $\bar{N}_t^C = \{N_t^1, \dots, N_t^K\} = \bar{m} = \{m_1, \dots, m_K\}, m_k = \text{Sup}\{j | \tau_j^k \leq t\}, k = 1, \dots, K$  : Vector of default level state of the children. Note that in our model setting,  $N_t^P = n = \sum_{k=1}^K m_k$

- $P(n, \vec{m}, \lambda_t, t) = P(N_t^P = n, \vec{N}_t^C = \vec{m}, \lambda_t, t)$ : The joint probability density of stochastic intensity process with  $\vec{N}_t^C$  defaults among the children and at intensity  $\lambda_t$ .
- $P(n, \vec{m}, t) = P(N_t^P = n, \vec{N}_t^C = \vec{m}, t)$ : The joint probability density of  $n$  defaults in the parent portfolio and  $\vec{m} = (m_1, \dots, m_k)$  defaults among the children.
- $\mu^k(m_k, t)$ : The NTD intensity in child  $k$  conditional on  $m_k$  prior defaults.
- $D(T) = \text{Exp}\left[-\int_0^T r(s)ds\right]$ : The risk-free discount factor from  $T$ .

## 2.2 Model Framework

We provide a dynamic, consistent, and arbitrage free valuation model for pricing tranches referencing a bespoke portfolio. To this end, we assume that the reference bespoke portfolio has a child-parent relation with one or more index portfolios. The indices are assumed to have no name overlapping among each other. We refer the overall portfolio that contains all the credits as the parent portfolio or simply the parent, and the sub-portfolios as children.

We consider two fundamental cases. In the first case, the parent is the bespoke portfolio that contains one or more index portfolios as sub-portfolios, perhaps plus the remaining names not belonging to any of the indices. In this case, the parent and the children are determined by the bespoke portfolio and the available indices. For example, if we want to price tranches referencing names in CDX and iTraxx, the parent is the union of the CDX and iTraxx indices.<sup>3</sup> The two natural children are the CDX and the iTraxx. In the second case, the bespoke portfolio is a sub-portfolio of an index. More complicated case can be constructed by combining these two.

Suppose that the parent contains  $N^P$  obligors, each obligor with constant recovery rate and notional 1. Each default then results in a parent portfolio loss of  $(1 - R)$ . Our multi-portfolio model can conveniently handle heterogeneous recovery or notional. The

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<sup>3</sup> Halperin (2009) proposed an implied multi-factor model for bespoke CDO tranche pricing. He provided such an example.

only requirement is for the recovery and notional to be uniform within each child. Treatment of heterogeneous recovery or heterogeneous notional is discussed in section 2.9.

We divide the parent into non-overlapping children. We assume both the parent and the children to be homogeneous. The recovery rate and notional amount of a credit are the same whether the credit is viewed as a member of the parent or as a member of the child to which the credit belongs. However, the spread of a credit when viewed as a member of the parent is different from that when viewed as a member of the child. The spread when viewed from the parent is equal to the spread of the parent, and that when viewed from the child is equal to the spread of the child. Since the child contains only a sub-set of the parent, the spread of the child differs from that of the parent. The dependence of spread on the underlying credit's portfolio association is a natural consequence of the top-down model assumption.

The above model setting assumes that the default is driven by the parent default intensity, and the default in the parent is distributed among the children according to the child conditional default probability. Therefore, our model is comprised of a default intensity process for the parent and a distribution of conditional probability of default among the children.

**Remark 2.2.1:** For simplicity, we assume a deterministic recovery rate. We recognize that a stochastic recovery rate or a time dependent deterministic recover rate may be required to achieve better fitting to the market prices when the market is under stress. Recent "credit crunch" has resulted in significant increase in the prices of the senior and super-senior index tranches. For example, the 60-100% tranche, which has no value under the customary assumption of 40% recovery, was recently trading with significant value [Amraoui and Hitier, 2008]. Our model framework can easily incorporate stochastic recovery rate assumption.

### 2.3. The Parent Default Intensity Process

The default intensity process of the parent portfolio in a top-down Markovian contagion default model can be generally expressed as<sup>4</sup>

$$d\lambda_t = \mu(\lambda_t, N_t^P, t)dt + \sigma(\lambda_t, N_t^P, t)dW_t + \xi(\lambda_t, N_t^P, t)dN_t^P \quad (1)$$

where  $dW_t$  is the standard Weiner process and  $N_t^P$  is the parent portfolio default level

$$N_t^P = \sum_{k=1}^N 1(\tau_k^P \leq t) \Rightarrow dN_t^P = N_t^P - N_{t-}^P = \sum_{k=1}^N [1(\tau_k^P \leq t) - 1(\tau_k^P < t)] \quad (2)$$

where  $1(x)$  is the indicator function and  $\tau_k^P$  is the k-th default time in the parent portfolio.

There are a number of top-down Markovian contagion models in the literature. Below, we mention four such models, representing distinct model features, numerical methods, and practical applications.

#### 2.3.1. Affine Intensity Model

Errais et al. (2009) proposed a class of affine intensity models with general formalization. In this paper, we mention one particular affine models which is formalized as<sup>5</sup>

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + \delta\xi dN_t^P \quad (3)$$

where  $\kappa, \lambda_\infty, \sigma, \delta$  are constant and  $\xi$  is the loss-given-default (LGD) that can be defined either as a random variable following an independent distribution or as a constant.

A stochastic LGD provides additional parameters that generally results in a better fit to the market tranche prices at the expense of introducing another dimension.  $\xi$  can also be specified as a function of default time  $\tau_k^P$  to introduce more parameters for potentially better fitting. One simple specification of random recovery is the uniform distribution [Errai et al., 2009].

<sup>4</sup> The BSLP model and the zero-volatility intensity model directly specify  $\lambda_t$ .

<sup>5</sup> The class of affine model described in Errais et al. (2009) is more general than Eq. (3). Here, we use the model (3) for exposition purpose because the emphasis of this paper is on the multi-portfolio framework rather than the particular form of the parent default model.

The affine model defined by Eq. (3) is self-affecting where the last term in Eq. (3) causes the intensity to jump upon a new default, making future default more likely. When there is no further default, the intensity diffuses and mean-reverts to the long term equilibrium level  $\lambda_\infty$ . Between defaults, the affine intensity model is similar to the CIR interest rate model which guarantees that the intensity can never become negative.

Affine intensity model (3) can be solved either by Monte Carlo simulation method [Giesecke and Kim, 2007] or by characteristic function method [Errais et al. 2009; Giesecke, 2007] adapted to the multi-portfolio setting. Monte Carlo method under the multi-portfolio model requires a two-step simulation procedure. In the first step, the parent default is simulated where the distribution is governed by Eq. (3). The algorithm of Giesecke and Kim (2007) can be used in the first step. If a default is encountered, the second step simulation, which can be considered as an inner simulation, determines which child the defaulter belongs. The distribution of the second step is governed by the conditional multi-portfolio default probability  $C_k(n, \bar{m}, t)$ . Monte Carlo simulation generally requires daily time step to account for the possibility of default clustering (default in consecutive days).

### **2.3.2. The Lopatin and Misirpashaev Model**

Assuming constant recovery rate across the parent, the Lopatin and Misirpashaev (2007) model is parameterized as

$$d\lambda_i = \kappa(\rho(N_i^P, t) - \lambda_i)dt + \sigma\sqrt{\lambda_i}dW_i \quad (4)$$

Instead of feeding the default back to the intensity process through the explicit linear term  $dN_i^P$  as done in the affine model, the credit default contagion effect is achieved by making the mean-reversion level  $\rho(N_i^P, t)$  depending on the parent default level  $N_i^P$ . The higher the default level is, the higher the mean-reversion level becomes which forces the spread to reach a higher equilibrium level, and hence making default more likely. The diffusion term produces credit spread volatility between defaults and provides the extra degrees of freedom for pricing dynamic products.

This model does not have a transform based solution due to the mean-reversion function. It can be solved by solving the Kolmogorov forward equation (15) for the joint default level density function  $P(N_t^P, \bar{N}_t^C, \lambda, t)$  which is defined below. Since Eq. (15) is a multi-portfolio extension of Eq. (2) in Lopatin and Misirpashaev (2007) model for single-portfolio, the numerical scheme can be adapted to solve Eq. (15). The Monte Carlo method outlined in the previous above for the affine model can be used for this model.

### 2.3.3. The BSLP Model

The intensity process of the BSLP model proposed by Arnsdorf and Halperin (2007) is

$$\lambda(N_t^P, Y_t, t) = (N - N_t^P) F(N_t^P, t) Y_t \quad (5)$$

where the contagion factor  $F(N_t^P, t)$  is a deterministic bilinear function of the default level and time. The random factor  $Y_t$  accounts for the spread volatility between defaults, and is assumed to follow a mean-reversion jump-diffusion model. Furthermore,  $Y_t$  is assumed to be dependent on  $dN_t^P$  but not the default level  $N_t^P$  itself which enables to calculate the conditional probability  $P(Y_{t+\Delta t} | N_{t+\Delta t}^P, N_t^P, Y_t)$  using a large time step  $\Delta t$ . In the BSLP model, the burden of calibration and simulation of the default intensity  $\lambda_t$  is shifted to the stochastic factor  $Y_t$  which is usually much simpler than  $\lambda_t$ . The BSLP model can be further simplified if  $Y_t$  is independent of the default level and is only a diffusion process in which case the diffusion process  $Y_t$  and the default level  $N_t^P$  are decoupled. The BSLP model can be solved by tree method with large time step.

### 2.3.4. Zero-Volatility Intensity Model

The zero-volatility intensity model is obtained by letting  $Y_t \equiv 1$  in Eq. (5), so it can be considered as a special case of the BSLP model. It is also called default only model since the random intensity change comes entirely from the jump in default level. Between defaults, the intensity remains deterministic. It can be specified as

$$\lambda_t(N_t^P, t) = (N - N_t^P) \sum_{k=0}^{N_t^P} \xi(k, t) \quad (6)$$

where  $\xi(k, t)$  is a piecewise constant function of  $k$  and  $t$ ,

$$\xi(k, t) = \xi_{i,j}, \text{ where } k \in (k_{i-1}, k_i], t \in (T_{j-1}, T_j] \quad (7)$$

In practice, the choices of the spatial node  $k_i$  and the temporal nodes  $T_j$  are problem dependent and may be less obvious than the single-portfolio case. In the single-portfolio case,  $k_i$  and  $T_j$  are usually chosen to coincide with the detachment points and maturity of the standard tranches. For example, we have  $\{k_i\}_{iTraxx} = \{0\%, 3\%, 6\%, 9\%, 12\%, 22\%\}$ , and  $\{k_i\}_{CDX} = \{0\%, 3\%, 7\%, 10\%, 15\%, 30\%\}$ . It is less obvious when the bespoke portfolio contains both iTraxx and CDX because the detachment points for iTraxx tranches and CDX tranches are different.

Since  $\xi(k, t)$  is positive, the intensity  $\lambda(N_t^P, t)$  is a non-decreasing function of default level  $N_t^P$ , accounting for the default contagion effect. It is easy to see from Eq. (6) that the credit spread is constant between defaults. Herbertsson shows that this model is capable of perfectly reproducing the market prices of the standard tranches. A distinct advantage of this model is that it enables analytic tranche pricing formula due to the constant Markov generator matrix.

The zero-volatility is sufficient for tranche pricing. Its numerical implementation is the simplest among the four models described above. For single-portfolio [Herbertsson, 2008] and two-portfolio cases, the matrix exponential method can be used [Moler and Loan, 2003]. A matrix exponential solution for a two-child case is presented in Appendix. For practical purpose, the number of children is likely limited to three with the bespoke portfolio being one of the children.

Below, we will demonstrate that the zero-volatility default model is not suitable for pricing spread volatility sensitive products such as option on tranche and forward tranche

because these products require sampling multiple points in time, and hence termed dynamic products.

Finally, we note that any of the above models can be used as the parent default intensity model. The choice depends on the model behavior and the ease of implementation and computation efficiency.

#### 2.4. The Child Portfolio Conditional Default Probability

Suppose that the parent is divided into  $K$  non-overlapping children, and child  $k$  has  $M_k$  initial obligors. Denote the default state of the children by  $\vec{N}_t^C = \vec{m} = (m_1, \dots, m_K)$  where  $m_k$  is the default level in child  $k$ . It is clear that the parent default level must be equal to the sum of all children's default levels,

$$n = \sum_{k=1}^K m_k \quad \text{and} \quad 0 \leq m_k \leq M_k \quad (8)$$

Define the conditional child default probability to be the probability that the next default is in child  $k$  conditional on the current default state  $\vec{m}$

$$C_k(n, \vec{m}, t) = P\left(\tau_{n+1}^P = \tau_{m_k+1}^k \mid \tau_n^P \leq t, \tau_{m_k}^k \leq t, \mathcal{F}_t^P \cup \left(\bigcup_{j=1}^K \mathcal{F}_t^j\right)\right) \quad (9)$$

then the child conditional default probability must satisfy

$$C_k(n, \vec{m}, t) \geq 0 \quad \text{and} \quad \sum_{k=1}^K C_k(n, \vec{m}, t) = 1 \quad (10)$$

The explicit dependency on  $\vec{m}$  emphasizes that the conditional child default probability  $C_k$  depends on the default state of all children, not just the default level of child  $k$ . We restrict  $C_k(n, \vec{m}, t)$  to be dependent only on the joint default state  $\vec{m}$  and time  $t$ . This implies that the child conditional default probability is constant between defaults. The spread volatility is due to the diffusion in the parent default intensity. We feel that introducing diffusion in  $C_k(n, \vec{m}, t)$  would intractable.

It is obvious that when  $m_k = M_k$ , all credits in child  $k$  have defaulted. The next default cannot be in child  $k$ . So the probability that next default is in child  $k$  is zero. On the other hand, if all credits in children other than child  $k$  have defaulted, the next defaulter must be in child  $k$ . When all surviving credits are in child  $k$ , we have  $N - n = M_k - m_k$ . Therefore, the conditional default probability must also satisfy

$$C_k(n, \vec{m}, t)|_{m_k=M_k} = 0 \quad \text{and} \quad C_k(n, \vec{m}, t)|_{m_k=M_k-N+n} = 1 \quad (11)$$

A specification of child conditional default probability that satisfies constraints (10) and (11) is

$$C_k(n, \vec{m}, t) = \frac{x_k g_k(x_k)}{\sum_{j=1}^K x_j g_j(x_j)} \quad (12)$$

where

$$x_k = (M_k - m_k)/(N - n) \quad (13)$$

and  $g_k(x)$  is some positive, deterministic function satisfying  $g_k(1) = 1$ .  $x_k = 1$  is equivalent to  $N - n = M_k - m_k$  and  $x_k = 0$  corresponds to  $m_k = M_k$ . For model consistency and symmetry, namely, all children must be treated the same way,  $g_k(x)$  should have the same functional form for all  $k$ . However, the parameters of  $g_k(x)$  can be different to reflect the difference in the average spreads of the children. In other words, we can specify  $g_k(x) = g(x; \xi_k)$  where  $\xi_k$  is a (time-dependent) parameter (potentially a vector) specifically for child  $k$ .

Eqs. (12) and (13) are a general specification of the child conditional default probability that must satisfy the conditions (10) and (11). However, we still have freedom to choose function  $g(x; \xi)$  which can conceivably depend on application. The simplest form for  $g_k(x)$  is a linear function,

$$g_k(x_k) = \xi_k(t) + x_k [1 - \xi_k(t)] \quad (14)$$

where  $\xi_k(t)$  is a deterministic function of time. In practice,  $\xi_k(t)$  is either piecewise constant or piecewise linear and must be fitted to the market data.

Although the specification (14) is the simplest function form under the framework of formulas (12) and (13), it nonetheless has some interesting features.

First, since  $m_k \leq M_k$  and  $n = |\bar{m}|$ , we see, by virtue of Eq. (13), that  $x_k \leq 1$  for all  $k \leq K$ . Furthermore, if a default occurs in the child  $k$ , then  $x_k$  decreases and  $x_{j \neq k}$  increases.

Second,  $\xi_k$  represents the (average) default risk level of child  $k$  relative to the (average) default risk level of the parent. If  $\xi_k = 1$ , we have  $g_k(x_k) = 1$  and  $C_k = 1/K$ .

This means that the parent default is uniformly distributed among the children. This is a trivial case where all children have the same aggregate riskiness.

Third,  $g_k(x_k)$  is an increasing function of the parameter  $\xi_k$ . If child  $k$  is riskier than the parent, we have  $\xi_k(t) > 1$  other thing being equal. On the other hand, if child  $k$  is less riskier than the parent, then  $\xi_k(t) < 1$ .

Fourth,  $g_k(x_k)$  is an increasing (decreasing) function of  $x_k$  if  $\xi_k(t) < 1$  ( $\xi_k(t) > 1$ ). However, the monotonicity on  $g_k(x_k)$  is not shared by  $C_k(n, \bar{m}, t)$ . Numerical experiments confirm that a default by the child  $k$  results in a decrease in  $C_k(n, \bar{m}, t)$  and an increase in  $C_j(n, \bar{m}, t)$  for all  $j \neq k$ . This indicates that when child  $k$  experiences a default, the likelihood that the next default is also in child  $k$  decreases. Since the sum of the child conditional default probabilities must be equal to 1, the probability that the next default being in another child increases. In other words, the linear model (14) implies conditional anti-default clustering for a child. However, it is important to distinguish this feature from the default clustering implied by the parent default contagion model.

Finally, it is important to note that  $g_k(x_k), k = 1, \dots, K$  must be calibrated simultaneously to the market data. In the non-trivial case, the parent portfolio is divided into two or more sub-portfolios. This means that  $\xi_k(t)$  are determined together.

**Remark 2.4.1:** If the children are identical in the sense that they have the same intrinsic spread and contain the same number of credits,  $C_k(n, \bar{m}, t)$  must be the same across children, because in this case one child cannot be riskier than another. Eqs. (12), (13) and (14) suggest that  $\xi_k(t)$  must be the same for all  $k$ . Setting  $\xi_k(t) = \xi$  in Eq. (14), we conclude that the only possibility is  $\xi = 1$  and  $C_k(n, \bar{m}, t) = x_k$ .

**Remark 2.4.2:** Single name default dynamics can be obtained by setting  $M_k = 1$ . In this case, we have  $x_k = (1 - 1(\tau^k < t)) / (N - n)$  where  $\tau^k$  is the default time of credit  $k$ . When the  $k$ -th credit defaults,  $m_k = 1$  and  $C_k(n, \bar{m}, t) = 0$ . We have  $\mu_k^{NTD}(m_k = 1, t) = 0$  (see Eq. (40) for definition). This means that the single name default intensity drops to zero after default. However, we need to track the defaulter identity through the child  $k$  natural filtration  $\mathcal{F}_t^k = \sigma\{\tau^k \leq t\}$ .

**Remark 2.4.3:** When there is only one child portfolio, i.e. the child is the same as the parent, we have  $m_1 = n$  and  $M_1 = N$ . This results in  $x_1 = 1$  and  $C_k \equiv 1$ . Hence, in this case the child default dynamics automatically degenerates to the parent process.

**Remark 2.4.4:** The dependence of  $C_k(n, \bar{m}, t)$  on  $n$  is technically redundant because of the constraint (8). Given the child default level state  $\vec{N}_t^C = \vec{m}$ , the parent default level  $N_t^P = n = |\vec{m}|$  is uniquely determined. Similarly, the unconditional multi-portfolio default probability density  $P(n, \bar{m}, \lambda, t)$  is uniquely determined by  $\vec{m}$ . Nevertheless, we will keep the  $N_t^P$  dependence explicit to emphasize the parent default level.

**Remark 2.4.5:** Eqs. (12) and (13) are a general form of the child conditional default probability based on the symmetry and boundary conditions. We are free to choose any particular form. Eq. (14) is the simplest specification.

**Remark 2.4.6:** The parameter  $\xi_k$  in (14) can be generalized to depend on the default level  $m_k$ . This provides additional freedom that may enable better fitting to the market price data.

### 2.5. Joint Multi-Portfolio Default Probability Density

Given the parent and child default counting processes  $(N_t^P, \bar{N}_t^C)$ , the joint unconditional default probability density  $P(n, \bar{m}, \lambda, t) = P(N_t^P = n, \bar{N}_t^C = \bar{m}, \lambda_t = \lambda, t)$  satisfies the Kolmogorov forward equation adapted to our multi-portfolio stochastic intensity process

$$\frac{\partial P(n, \bar{m}, \lambda, t)}{\partial t} = -\frac{\partial}{\partial \lambda} [\mu(\lambda, n, t)P(n, \bar{m}, \lambda, t)] + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [\sigma^2(\lambda, n, t)P(n, \bar{m}, \lambda, t)] + \lambda \sum_{k=1}^K P(n-1, \bar{l}_k, \lambda, t) C_k(n-1, \bar{l}_k, t) - \lambda P(n, \bar{m}, \lambda, t) \quad (15)$$

where  $\bar{l}_k = \{l_{j \neq k} = m_j, l_k = m_k - 1\}$  is the child default state that can reach the state  $\bar{m}$  in an infinitesimal time interval. Note that the joint default probability density is non-zero only if  $n = |\bar{m}|$  and  $n-1 = |\bar{l}_k|$ .

If the default intensity is zero, then there is no chance of default. Therefore, we impose the boundary condition

$$P(n, \bar{m}, 0, t) = 0 \quad (16)$$

Obviously, there can be no default at time  $t = 0$ . As a result, the initial condition is

$$P(n, \bar{m}, \lambda, 0) = f(\lambda) \times 1(n=0) \quad (17)$$

By definition, the stochastic default density function for the parent portfolio is

$$P(N_t^P = n, \lambda, t) = \sum_{|\bar{m}|=n} P(n, \bar{m}, \lambda, t) \quad (18)$$

where the summation is over all possible state  $\bar{m}$  satisfying  $|\bar{m}| = \sum_{k=1}^K m_k = n$ .

Because the default state  $\bar{m}$  can be reached only from the state  $\bar{l}_k, k = 1, \dots, K$ , over an infinitesimal time interval, we have

$$\sum_{|\bar{m}|=n} \sum_{k=1}^K P(n-1, \bar{l}_k, \lambda, t) C_k(n-1, \bar{l}_k, t) = P(n-1, \lambda, t) \quad (19)$$

Substituting Eqs. (17), (18) and (19) into Eq. (15), we obtain the Kolmogorov forward equation for the stochastic parent default density

$$\begin{aligned} \frac{\partial P(n, \lambda, t)}{\partial t} = & -\frac{\partial}{\partial \lambda} [\mu(\lambda, n, t)P(n, \lambda, t)] + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [\sigma^2(\lambda, n, t)P(n, \lambda, t)] \\ & + \lambda [P(n-1, \lambda, t) - P(n, \lambda, t)] \end{aligned} \quad (20)$$

Note that Eq. (20) is the same as the Eq. (2) in Lopatin and Misirpashaev (2007).

Pricing of credit derivatives entails the calculation of expectation of some payoffs which are generally functions of portfolio default loss but not the intensity. For constant recovery, loss is proportional to the default level. As a result, the payoff is a function of the portfolio default level. The model calibration involves fitting to the prices of standard tranche and index. The model may also need to be calibrated to option prices to determine the dynamic model parameters controlling the spread volatility and mean reversion.

Calibration to the current market prices of standard tranches and index requires only the probability density of joint default state which is independent of  $\lambda_t$ . Thus, it is possible to eliminate the  $\lambda_t$  dependence in this part of the calibration. To this end, we first note that the joint probability satisfies the following transition equation

$$\begin{aligned} P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C, \lambda_{t+\Delta t}, t + \Delta t) = & \sum_{N_t^P, \vec{N}_t^C} \int_0^\infty P(N_t^P, \vec{N}_t^C, \lambda, t) \times \\ & P(\lambda_{t+\Delta t} | N_{t+\Delta t}^P, N_t^P, \lambda) P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C | N_t^P, \vec{N}_t^C, \lambda) d\lambda \end{aligned} \quad (21)$$

where we have used the fact that the parent default intensity  $\lambda_t$  is independent of the child default level vector  $\vec{N}_t^C$ .

Integrating both sides of Eq. (21) with respect to  $\lambda_{t+\Delta t}$ , we obtain the unconditional joint multi-portfolio probability density of  $\vec{N}_{t+\Delta t}^C$  at time  $t + \Delta t$ ,

$$P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C, t + \Delta t) = \sum_{N_t^P, \vec{N}_t^C} \int_0^\infty P(N_t^P, \vec{N}_t^C, \lambda, t) P(N_{t+\Delta t}^P, \vec{N}_{t+\Delta t}^C | N_t^P, \vec{N}_t^C, \lambda) d\lambda \quad (22)$$

where the unconditional joint default probability density is defined by

$$P(N_t^P, \vec{N}_t^C, t) = \int_0^\infty P(N_t^P, \vec{N}_t^C, \lambda, t) d\lambda \quad (23)$$

Making the usual assumption of no simultaneous defaults, the conditional probability of next defaulter being in child k is

$$P(N_{t+\Delta t}^P = n, \bar{N}_{t+\Delta t}^C = \bar{m} | N_t^P = n-1, \bar{N}_t^C = \bar{l}_k, \lambda_t) = \lambda_t (n-1) \Delta t C_k(n-1, \bar{l}_k) \quad (24)$$

Substituting (24) into (21), setting  $N_{t+\Delta t}^P = n, \bar{N}_{t+\Delta t}^C = \bar{m}$ , and using the relation (note that  $\lambda_t$  is independent of the child default level  $\bar{N}_t^C$ ),

$$\int_0^\infty \lambda P(N_t^P, \bar{N}_t^C, \lambda, t) d\lambda = P(N_t^P, \bar{N}_t^C, t) E(\lambda_t | N_t^P) \quad (25)$$

we obtain the forward differential equation for the unconditional joint child default probability density  $P(n, \bar{m}, t)$ <sup>6</sup> (dependence on  $n$  is redundant and only for clarity)

$$\frac{dP(n, \bar{m}, t)}{dt} = -G(n, t)P(n, \bar{m}, t) + G(n-1, t) \sum_{k=1}^K C_k(n-1, \bar{l}_k, t) P(n-1, \bar{l}_k, t) \quad (26)$$

where  $\bar{l}_k = \{l_{j \neq k} = m_j, l_k = m_k - 1\}$  and

$$G(n, t) = E(\lambda_t | N_t^P = n) = \int_0^\infty \lambda P(N_t^P = n, \lambda, t) d\lambda / P(N_t^P = n, t) \quad (27)$$

$$P(N_t^P = n, t) = \int_0^\infty P(N_t^P = n, \lambda, t) d\lambda$$

The initial condition for Eq. (26) that is consistent with Eq. (17) is

$$P(n, \bar{m}, 0) = \int_0^\infty f(\lambda) d\lambda \times 1(n=0) = \delta_{n,0} \quad (28)$$

It can be shown that the unconditional parent default density satisfies the equation<sup>7</sup>

$$\frac{dP(n, t)}{dt} = -G(n, t)P(n, t) + G(n-1, t)P(n-1, t) \quad (29)$$

where  $P(n, t) = \sum_{|\bar{m}|=n} P(n, \bar{m}, t)$  with  $|\bar{m}| = \sum_{k=1}^K m_k$ .

<sup>6</sup> Eq. (26) can also be derived by integrating Eq. (15) with respect to  $\lambda$  and by using Eq. (27).

<sup>7</sup> Eq. (29) is the same as Eq. (6) in Lopatin and Misirpashaev (2007).

Lopatin and Misirpashaev (2007) called the function  $G(n,t)$  the local intensity (LI) because it is the expected intensity conditional on the default level. They extend the method of Markovian projection, a term first used by Piterbarg (2007) for fast calibration of stochastic volatility interest rate model but the method was used earlier by Dupire (1994) for the local volatility model for equity derivative, to the credit derivative modeling. If the stochastic intensity  $\lambda_t$  is self-affecting, a higher level of default will on average result in a higher level of default intensity. Therefore, the local intensity  $G(n,t)$  will be an increasing function of  $n$  under the default contagion models.

**Remark 2.5.1:** By deriving the local intensity model equation (26), we have effectively proven that  $G(N_t^P, t)$  is the local intensity of the original stochastic intensity model, in the sense that the local intensity is the conditional expectation of the stochastic intensity.

**Remark 2.5.2:** The local intensity  $G(n,t)$  is implicitly conditioned on today's filtration  $\mathcal{F}_0^P$ . The Gyongy theorem, extended to jump process, implies that the SI model and the LI model have the same marginal density conditional on today's information. Both models are equivalent for pricing static products, but they generally differ for pricing dynamic products that depend on future filtration. We will elaborate in Sec. 2.8.

**Remark 2.5.3:** If there is only one child, then the child must be identical to the parent. In this case, we must have  $m = n$ ,  $C(n,m,t) = 1$  and  $P(n,m,t) = P(n,t)$ . Eqs. (15) and (26) reduce to the Kolmogorov forward equation under the single-portfolio setting.

### 2.6. The Local Intensity $G(N_t^P, t)$

Eq. (26) indicates that the initial condition for the local intensity default density  $P(n, \bar{m}, t)$  must be consistent with the initial condition Eq. (17) for the stochastic intensity default probability density. On the one hand, the definition  $G(n,t) = E(\lambda_t | N_t^P = n)$  suggests that  $G(n > 0, t = 0) = 0$  since  $P(n > 0, 0) = 0$ . On the other hand, if we take  $G(n,0)$  as the

limit  $G(n > 0, 0) = \lim_{t \rightarrow 0} G(n > 0, t)$ , we will see below that it not possible for  $G(n > 0, t = 0) = 0$ .

By definition,

$$G(n = 0, 0) = \lim_{t \rightarrow 0} \int_0^{\infty} \lambda P(n = 0, \lambda, t) d\lambda / P(n = 0, t) = \int_0^{\infty} \lambda f(\lambda) d\lambda / \int_0^{\infty} f(\lambda) d\lambda \quad (30)$$

If  $f(\lambda) = \delta(\lambda - \lambda_0)$ , then  $G(0, 0) = \lambda_0$ . On the other hand, the zero-volatility model leads to  $G(n = 0, t = 0) = E(\lambda_0 | N_0^P = 0) = \lambda_0$ , the initial stochastic intensity.

The small time asymptotic behavior of  $P(N_t^P, \lambda_t, t)$  is dominated by the jump behavior. Neglecting the diffusion effect, Eq. (20) becomes

$$\frac{dP(n, \lambda, t)}{dt} = \lambda P(n - 1, \lambda, t) - \lambda P(n, \lambda, t), \quad P(n, \lambda, 0) = f(\lambda) \mathbb{1}(n = 0) \quad (31)$$

Thus the small time approximate solution is

$$P(n, \lambda, t) = f(\lambda) \times (\lambda t)^n e^{-\lambda t} / n! \quad (32)$$

This leads to the initial condition for the local intensity function

$$G(n, 0) = \lim_{t \rightarrow 0} \int_0^{\infty} \lambda P(n, \lambda, t) d\lambda / P(n, t) = \int_0^{\infty} \lambda^{n+1} f(\lambda) d\lambda / \int_0^{\infty} \lambda^n f(\lambda) d\lambda \quad (33)$$

Eq. (33) shows that  $G(n > 0, 0) \neq 0$ .

The actual computation of the local intensity function  $G(n, t)$  depends on the choice of the parent default model. In the following, we summarize the calculation of local intensity function under the four models described in Sec. 2.3.

### 2.6.1. The Lopatin and Misirpashaev Model

In the Lopatin and Misirpashaev model (2007),  $\lambda_t$  is given Eq. (4), thus

$$E(\lambda_t | N_t^P) = \Lambda(N_t^P, t) \quad (34)$$

which is the Eq. (8) in Lopatin et al. (2007). The calculation of the local intensity function  $\Lambda(N_t^P, t)$  follows the procedure described in their paper. It is clear that Eq. (26) is the multi-portfolio extension of the local intensity equation for single portfolio.

### 2.6.2. Affine Model

In the affine intensity model (3), we can use the Fourier transform method [Giesecke, 2007; Ding et al. 2008]. Define the joint characteristic function

$$\Phi(u, \Psi(v), t) = E\left(e^{iu\lambda_t + ivN_t}\right) = \sum_{k=0}^N E\left(e^{iu\lambda_t} \mid N_t^P = k\right) [1 - \Psi(v)]^k P(N_t^P = k) \quad (35)$$

where  $\Psi(v) = 1 - e^{iv}$ , we deduce that

$$E(\lambda_t \mid N_t^P = n) P(N_t^P = n) = -i \left\{ \frac{\partial^{n+1} \Phi(u, w)}{\partial u \partial w^n} \Big|_{u=0, w=1} \right\} \div \left\{ \frac{\partial^n \Phi(u, w)}{\partial w^n} \Big|_{u=0, w=1} \right\} \quad (36)$$

provided that the partial derivatives exist.

The Fourier transform method can be applied to any intensity model for which the characteristic function and its derivatives exist.

### 2.6.3. The BSLP Model

Under the BSLP model framework (5),

$$E(\lambda_t \mid N_t^P) = F(N_t^P, t) E(Y_t \mid N_t^P) \quad (37)$$

So the burden of calculating  $E(\lambda_t \mid N_t^P)$  is shifted to calculating  $E(Y_t \mid N_t^P)$ . If  $Y_t$  is independent of the default level increment, i.e.  $\gamma(Y_t, t) = 0$  in Eq. (18) of Arnsdorf and Halperin (2008), Eq. (37) essentially reduces to calculating  $E(Y_t \mid N_t^P) = E(Y_t)$ . In practice,  $E(Y_t)$  is much simpler to compute than  $E(Y_t \mid N_t^P)$ .

### 2.6.4. Zero-Volatility Intensity Model

As described in Sect. 2.3.4, the zero-volatility intensity is a deterministic function of the default level and time. Hence, we have

$$E(\lambda_t \mid N_t^P) = \lambda(N_t^P, t) = (N - N_t^P) \sum_{k=0}^{N_t^P} \xi(k, t) \quad (38)$$

In the zero-volatility intensity model, the intensity depends only on the default level. So the local intensity is the same as the original stochastic intensity.

### 2.7. The Child Default Intensity Process

The conditional probability that the next default is in child  $k$  given  $n$  total prior defaults in the parent and  $m_k$  prior defaults in child  $k$  is given by

$$C^k(n, m_k, \lambda_t, t) = P(\tau_{n+1}^P = \tau_{m_k+1}^k \mid \tau_n^P < t, F_t^P \cup F_t^k) \\ = \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) C_k(n, \bar{m}, t) / \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) \quad (39)$$

where  $\mathfrak{S}_k(n, i) = \{\bar{m} \mid |\bar{m}| = n; m_k = i; 0 \leq m_j \leq M_j, j = 1, \dots, K\}$  is the set of child default state subjected to the constraint that the parent default count is  $n$  and child  $k$  default count is  $m_k = i$ .

The next-to-default intensity of child  $k$ , conditional on  $m_k$  prior defaults is

$$\mu^k(m_k, t) = \sum_{n=m_k}^{N-M_k+m_k} \lambda(n, t) C^k(n, m_k, t) \quad (40)$$

which can be used to generate default paths for child  $k$ . It is clear that by construction, we have  $\mu^k(m_k = M_k, t) = 0$ , meaning that once all credits in child  $k$  have defaulted, its default intensity drops to zero.

The unconditional probability of  $n$  parent defaults and  $m_k$  child  $k$  defaults is

$$P^k(n, m_k, \lambda_t, t) = \sum_{\bar{m} \in \mathfrak{S}_k(n, m_k)} P(n, \bar{m}, \lambda_t, t) \quad (41)$$

and the unconditional probability of  $m_k$  defaults in child  $k$  is

$$P^k(m_k, \lambda_t, t) = \sum_{n=m_k}^{N-M_k+m_k} P^k(n, m_k, \lambda_t, t) \quad (42)$$

Eq. (42) can be used to calculate expected loss for child  $k$ , and the expected loss of tranches referencing child  $k$ .

### 2.8. Local Intensity Model and Stochastic Intensity Model

It has been reported that tranche pricing is the same under the LI model and the SI model, and that they result in different prices for forward starting tranches and tranche options

[Arnsdorf and Halperin, 2008; Lopatin and Misirpashaev, 2007]. In this section, we show that the LI model and the SI model are equivalent for pricing static products such as tranches using the Gyongy's theorem. But they are different for pricing dynamic products such as tranche option, forward starting and Leveraged-Super-Senior (LSS).

For simplicity and ease of notation, we use the single portfolio setting to explain the reason why the local intensity model is not equivalent to the stochastic intensity model. The multi-portfolio setting follows essentially the same line.

In Sec. 2.5, we showed that the LI process, which can be considered as the Markovian projection of the SI process [Piterbarg, 2007] and has the same marginal default density as the SI process according to the Gyongy's theorem, is conditioned on today's information filtration  $\mathcal{F}_0^F$ . The local intensity conditional on future filtration will be different. To prove this point, we express the local intensity conditioned on a filtration of future time  $s$  by

$$G(N_t^P, t; N_s^P, \lambda_s) = E\{\lambda_t | N_t^P, F_s^P\} = \int_0^\infty \lambda P(\lambda | N_t^P, N_s^P, \lambda_s) d\lambda = \int_0^\infty \lambda \frac{P(N_t^P, \lambda | N_s^P, \lambda_s)}{P(N_t^P | N_s^P, \lambda_s)} d\lambda \quad (43)$$

where  $t > s$  and  $F_s^P = \sigma\{\tau_k^P \leq s\} \cup \{\lambda_u | u \leq s\}$  is the time  $s$  filtration of the parent default and intensity and default.  $P(N_t^P, \lambda_t | N_s^P, \lambda_s)$  is the SI model probability density at time  $t$  conditional on  $\mathcal{F}_s^P$ .

In the zero-volatility intensity model, the default intensity depends on the current default level  $N_t^P$  only, and is independent of  $N_s^P, \lambda_s; s < t$ . As a result, we conclude that in the zero-volatility intensity model, the conditional local intensity and the (unconditional) local intensity are the same,

$$G^{Zero-Vol}(N_t^P, t; N_s^P, \lambda_s) = E\{\lambda_t | N_t^P, \tilde{F}_s^P\} = E\{\lambda_t | N_t^P\} = G^{Zero-Vol}(N_t^P, t) \quad (44)$$

Later in this section, we will show that Eq. (44) is the reason that the LI model and SI model are equivalent for the zero-volatility intensity model.

The LI model default level probability density conditional on the future time  $s$  is given by

$$P(N_t^P | N_s^P, \lambda_s) = \int_0^\infty P(N_t^P, \lambda | N_s^P, \lambda_s) d\lambda \quad (45)$$

By the law of iterative expectation, the local intensity seen today is equal to<sup>8</sup>

$$\begin{aligned} G(n, t) &= G(N_t^P = n, t; N_0^P = 0, \lambda_0) = E\{G(N_t^P, t; N_s^P, \lambda_s) | N_t^P = n, \tilde{F}_0^P\} \\ &= \sum_{m=0}^n \int_0^\infty G(N_t^P = n, t; N_s^P = m, \lambda_s) P(N_s^P = m, \lambda_s | N_t^P = n) d\lambda_s \end{aligned} \quad (46)$$

Eq. (43) shows that the time  $s$  local intensity is the expectation of the time  $t$  local intensity conditioned on the time  $s$  filtration  $\mathcal{F}_s^P$ . Eqs. (43) and (46) show that (a) the local intensity seen from a future time is not equivalent to the local intensity seen today, and (b) the today's local intensity is the expected future local intensity. This suggests that the LI model is equivalent to the SI model for pricing static products which depend only on today's information, but they are not equivalent when pricing products with payoff depending on two or more future times<sup>9</sup>.

Suppose a payoff function  $F(N_T^P, T)$  that depends on the default level at time  $T$  only. Assuming zero interest rate, the expected payoff is

$$E\{F(N_T^P, T) | F_0^P\} = \sum_{n=0}^N \int_0^\infty F(n, T) P(n, \lambda_T) d\lambda_T = \sum_{n=0}^N F(n, T) P(n, T) \quad (47)$$

where  $P(n, T)$  is the time  $t = 0$  local intensity probability density defined in Eq. (27).

<sup>8</sup> The today's local intensity is analogous to the Dupire's local volatility function for the equity model [Dupire, 1994].

<sup>9</sup> This is similar to equity derivative pricing where the LV and SV models are equivalent in terms of pricing vanilla European options, but differ for forward starting option. Piterbarg (2007) pointed out that the Markovian projection model, such as the LV model, differs from the original SV model in pricing of instruments that require sampling multiple points in time.

Hence, the expected payoff is the same under the LI model and SI model. Since the tranche price is just a weighted average of the expected payoff (47), we conclude that the LI and SI models are equivalent for tranche pricing.

Now, let's suppose that the payoff function  $H(N_T^P, N_U^P)$  depends on the default level at two future times  $T < U$ . Under the SI model, the expected payoff is

$$\begin{aligned} V^{SI}(T, U) &= E\{H(N_T^P, N_U^P)F_0^P\} = \sum_{n=0}^N \sum_{m=0}^n \iint H(m, n)P(m, n, \lambda_T, \lambda_U) d\lambda_T d\lambda_U \\ &= \sum_{n=0}^N \sum_{m=0}^n H(m, n)P^{SI}(n, U, m, T) \end{aligned} \quad (48)$$

where (the superscript SI denotes that the conditional probability density is under the SI model to differentiate with the same conditional probability  $P^{LI}(n|m)$  under the LI model. We use  $n$  for  $N_U^P$ , and  $m$  for  $N_T^P$  to simplify notation.)<sup>10</sup>

$$\begin{aligned} P^{SI}(n, U, m, T) &= \int_0^\infty P(n, U, m, \lambda_T, T) d\lambda_T = P^{SI}(n, U|m, T)P(m, T) \\ P^{SI}(n, U|m, T) &= \int_0^\infty P(m, \lambda_T)P(n, U|m, \lambda_T, T) d\lambda_T / P(m, T) \\ P(n, U|m, \lambda_T, T) &= \int_0^\infty P(n, \lambda_U|m, \lambda_T, T) d\lambda_U \end{aligned} \quad (49)$$

On the other hand, under the LI model, the expectation in Eq. (47) becomes

$$V^{LI}(T, U) = E\{H(N_T^P, N_U^P)\tilde{F}_0^P\} = \sum_{n=0}^N \sum_{m=0}^n H(m, n)P^{LI}(n, U, m, T) \quad (50)$$

Comparing expression (48) with (50), we see that for  $V^{SI}(T, U) = V^{LI}(T, U)$  to hold, we must have  $P^{LI}(n, U, m, T) = P^{SI}(n, U, m, T)$  for  $\forall((m, T), (n, U))$ . In the following, we will show that this is generally not the case.

Applying Eq. (29) to the time  $T$  filtration  $\mathcal{F}_T^P$ , we see that  $P(n, U|m, \lambda_T, T)$  is the solution to the “forward” local intensity model equation for time  $U > T$ ,

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<sup>10</sup> We sometime omit  $U$  and  $T$  under the implicit assumption that  $U$  is associated with  $n$  and  $T$  with  $m$ .

$$\begin{aligned} \frac{dP^{SI}(n,U|m,\lambda_T)}{dU} &= -G(n,U;m,\lambda_T)P^{SI}(n,U|m,\lambda_T) \\ &+ G(n-1,U;m,\lambda_T)P^{SI}(n-1,U|m,\lambda_T) \end{aligned} \quad (51)$$

where  $G(n,U;m,\lambda_T)$  is the local intensity conditional on  $\mathcal{F}_T^P$ , defined in Eq. (43).

Multiplying Eq. (51) by  $P(N_T^P, \lambda_T)$  and integrating the result over  $\lambda_T$ , we obtain the equation for the SI model joint default level probability density

$$\begin{aligned} \frac{dP^{SI}(n,U,m,T)}{dU} &= -P^{SI}(n,U,m,T) \int_0^\infty G(n,U;m,\lambda_T) \frac{P(n,U,m,\lambda_T)}{P^{SI}(n,U,m,T)} d\lambda_T \\ &+ P^{SI}(n-1,U,m,T) \int_0^\infty G(n-1,U;m,\lambda_T) \frac{P(n-1,U,m,\lambda_T)}{P^{SI}(n-1,U,m,T)} d\lambda_T \end{aligned} \quad (52)$$

Under the LI model, the conditional default probability density satisfies Eq. (29),

$$\frac{dP^{LI}(n,U|m,T)}{dU} = -G(n,U)P^{LI}(n,U|m,T) + G(n-1,U)P^{LI}(n-1,U|m,T) \quad (53)$$

Multiplying Eq. (53) by  $P(m,T)$  and noticing that  $P^{LI}(n,m) = P^{LI}(n|m)P(m)$ , we obtain the equation governing the LI model joint default level probability density

$$\frac{dP^{LI}(n,U,m,T)}{dU} = -G(n,U)P^{LI}(n,U,m,T) + G(n-1,U)P^{LI}(n-1,U,m,T) \quad (54)$$

Except for the zero-volatility intensity model, we obtain<sup>11</sup>

$$G(n,U) \neq \int_0^\infty G(n,U;m,\lambda_T) \frac{P(n,U,m,\lambda_T)}{P^{SI}(n,U,m,T)} d\lambda_T \quad (55)$$

This means that except for the zero-volatility intensity model, we have

$$P^{SI}(n,U,m,T) \neq P^{LI}(n,U,m,T) \quad (56)$$

which leads to

$$V^{LI}(T,U) \neq V^{SI}(T,U) \quad (57)$$

Therefore, the pricing of products whose payoffs depend on the default level at two or more future times is different under the LI model from that under the SI model if the

<sup>11</sup> Under the zero-volatility intensity model, Eq. (55) becomes Eq. (44).

SI model has a non-zero diffusion term (or equivalently, the probability density depends on  $\lambda_t$  in a non-trivial way). It will be shown in the next section that the forward starting tranche and tranche option fall into this category and hence have different pricing under the LI model and SI model, as has been confirmed by the numerical results presented in Arnsdorf and Halperin (2008) and Lopatin and Misirpashaev (2007).

**Remark 2.8.1:** It is helpful to understanding the difference between the LI and SI models for pricing dynamic products using equity derivative analogy. The LI model is analogous to the local volatility (LV) model in equity derivative pricing. Hagan et al. (2002) showed that the LV model, while having the same marginal distribution as SV model, predicts the implied volatility dynamics that is opposite of the behavior observed in the market. This suggests that LV model is not suitable for pricing products like the forward starting option for which the volatility dynamics is important. The implied volatility dynamics of the LV model is different from that of the SV model. However, both LV and SV model are calibrated to today's implied volatility surface. From the mathematical standpoint, the vanilla options are similar to tranches and the forward starting options are similar to the forward starting tranches. This is not to say that the credit and equity derivatives are the same. They are not. For one thing, the equity market exhibits a negative implied volatility skew where the implied volatility decreases as the strike increases. The tranche option has positive implied volatility skew where the equivalent Black implied volatility increases with increasing strike.

### **2.9. Heterogeneous Recovery Rate and Notional Amount**

In certain situation, we may wish to assign heterogeneous recovery rates to the credits in the bespoke portfolio. It has been well documented that the recovery rate decreases with increasing spread. In other words, the riskier are the credits, the lower the expected recovery rates would be. For example, Markit determined through dealer poll that the recovery rate is 40% for CDX.NA.IG and 30% for CDX.NA.HY. Therefore, if the bespoke portfolio contains credits from these two indices, the model will have to be calibrated to the CDX.NA.HY tranches and the CDX.NA.IG tranches such that the model

is consistent with these two index pricing. Consequently, we need to assign 40% recovery rate to the CDX.NA.IG credits and 30% recovery to the CDX.NA.HY credits. This can be conveniently accomplished within the multi-portfolio model framework. For this case, we assign 40% recovery to the child containing the CDX.NA.IG credits and 30% recovery to the child of CDX.NA.HY credits.

Similarly, heterogeneous notional amount can also be handled by the multi-portfolio model.

### **3. Product Pricing**

In this section, we outline the steps for pricing bespoke tranche, forward starting tranche and tranche option. The CDS tranches are sometimes called static products in the same sense of straight bonds or vanilla European equity options. Static credit products may be loosely defined as those instruments whose pricing is not affected by spread volatility. Products which cannot be classified as static are dynamic products. Under this definition, spread volatility must have non-negligible impact on the pricing of dynamic products. We note that static models have been used to price dynamic credit products such as forward starting tranche.<sup>12</sup> In the following, we use portfolio as the generic term for a bespoke portfolio which can be either the parent or a child in our multi-portfolio framework.

#### **3.1. CDS Tranche**

A CDS tranche is a contract between two parties where one party sells credit protection and the other buys protection on a specific portion in the capital structure of a pool of synthetic CDSs. Given a reference portfolio, a tranche is defined by the attachment point (AP) and the detachment point (DP), maturity and the contract spread<sup>13</sup>. The CDSs in the

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<sup>12</sup> In equity derivative modeling, the LV model is popular among practitioners for exotic options such as cliquet options, ladder options even though the LV model is known to under-price these products.

<sup>13</sup> Unlike cash CDO, synthetic CDO does not have payment waterfall. Payments on a tranche are solely determined according to the tranche loss which in turn depends entirely on the cumulative loss on the reference portfolio and the tranche attachment and detachment points.

reference portfolio are referred to as credit, reference obligation or obligor. The protection seller receives a periodic payment from the protection buyer based on the tranche's remaining notional amount and the tranche spread. In exchange for the coupon payments, the protection seller is obligated to make contingent payment to the protection buyer for the portfolio cumulative default loss exceeding the AP. The total contingent payment is capped at  $DP - AP$ .

Let  $L_t^{PF} = (1 - R)N_t^{PF} / N^{PF}$  be the cumulative percentage loss on the reference bespoke portfolio (PF), the tranche loss function at time  $t$  is defined as

$$L_t^{TR} = f(L_t^{PF}) = (L_t^{PF} - AP)^+ - (L_t^{PF} - DP)^+ \quad (58)$$

where the superscript PF denotes portfolio. Note that the portfolio itself is considered as the 0-100% tranche in which case we have  $L_t^{TR} = L_t^{PF}$ .

The tranche loss function (58) suggests that CDO tranche can be viewed as a long call option on the cumulative portfolio loss with the strike AP and a short call option with the strike DP. Using the equity derivative terminology, the tranche can be viewed as a bull call spread on the cumulative portfolio loss.

Assuming independence between the interest rate and the default level, and neglecting accrued premium, the mark-to-market value (MTM) to the tranche protection seller is

$$\begin{aligned} MTM_{Tranche} = & P_{Upfront} + S \sum_{k=1}^K D(T_k) \{ DP - AP - E_t [R(T_k) + L_{T_k}^{TR}] \} \Delta T_k \\ & - \sum_{k=1}^K \frac{1}{2} [D(T_{k-1}) + D(T_k)] E_t (L_{T_k}^{TR} - L_{T_{k-1}}^{TR}) \end{aligned} \quad (59)$$

In Eq. (59),  $R(t) = [RN_t^{PF} / N^{PF} - (1 - DP)]^+$  is the amortizing amount removed from (the top of) the portfolio. Clearly,  $R(t)$  can be non-zero only for tranches with  $DP > 1 - R$ .<sup>14</sup> So it affects only the most senior tranche. If the tranche is identical to the

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<sup>14</sup> If the recovery rate is heterogeneous, R is the recovery rate associated with the child portfolio the tranche references.

portfolio, i.e. AP = 0 and DP = 100%, we have  $R(T) + L_T^{TR} = N_T^{PF} / N^{PF}$ , the portfolio notional loss due to default. In Eq. (59),  $P_{Upfront}$  is the upfront price quote and S is the tranche running spread. The market convention for investment grade (IG) is to quote equity and junior mezzanine tranches by an upfront price plus a fixed running spread (usually 500 bps), and to quote mezzanine and senior tranches in all running spread form. Emerging market (EM) and High Yield (HY) indices quote prices in upfront with no running spread. Compared with the all running spread tranche, upfront payment reduces the spread sensitivity (DV01) and P&L convexity.

Eqs. (58) and (59) show that tranche pricing essential involves the calculation of

$$\begin{aligned} E(L_T^{PF} - A)^+ &= E[(1 - R)N_T^{PF} / N^{PF} - A]^+ \\ &= \sum_{\vec{N}_T^C} [(1 - R)N_T^{PF} / N^{PF} - A]^+ P(N_T^P, \vec{N}_T^C, T) \end{aligned} \quad (60)$$

with

$$P(N_T^P, \vec{N}_T^C, T) = \int_0^\infty P(N_T^P, \vec{N}_T^C, \lambda_T, T) d\lambda_T.$$

The multi-portfolio default level density  $P(N_T^P, \vec{N}_T^C, T)$  can be obtained by solving Eq. (26). Eq. (60) shows that tranche pricing essentially requires only the marginal default level distribution. In other words, the diffusion component in the stochastic intensity model does not matter. As a result, the zero-volatility model is sufficient. Since the diffusion is irrelevant in tranche pricing, the diffusion volatility parameters in the SI model cannot be determined from calibrating to the tranche prices alone. These observations have been previously pointed out by Arnsdorf and Halperin (2008), Lopatin and Misirpashaev (2007).

The fact that tranche pricing does not require intensity volatility suggests that it is more efficient to fit tranche prices using the distribution density  $P(N_t^P, \vec{N}_t^C, t)$  of Markovian projection model rather than the original full blown stochastic intensity model.

### 3.2. Forward Starting Tranche

A forward starting tranche is a forward contract on a tranche that starts at a future date with predetermined notional amount, maturity, reference portfolio, attachment and detachment points, and tranche premium. At the tranche starting date, the AP and DP are adjusted according to the portfolio loss such that the tranche subordination remains unchanged. Suppose that the AP and DP for the forward starting tranche are  $a$  and  $b$ , and the forward starting date is  $T_0$ . If the cumulative portfolio loss at  $T_0$  is  $L_{T_0}^{PF}$ , the AP and DP of the forward starting tranche at the starting date  $T_0$  are  $A = a + L_{T_0}^{PF}$  and  $B = b + L_{T_0}^{PF}$ . The effective tranche subordination is still  $a$  and the effective tranche size remains to be  $b - a$ . Both  $A$  and  $B$  are capped at 100%. After  $T_0$ , it is a regular tranche.

Let the forward starting tranche maturity be  $T_N$ , The MTM value before the starting date  $t < T_0$  to the protection seller is equal to the present value of the difference over the life of the tranche between the expected tranche loss and the expected premium payment. Assuming unit notional amount and option contract spread  $S$ , this is given by<sup>15</sup>

$$MTM(t) = S \sum_{k=1}^N E_t(L_{T_k}^{TR} - L_{T_{k-1}}^{TR})D_{T_k} - \sum_{k=1}^N E_t(b - a - L_{T_{k-1}}^{TR})D_{T_k} \quad (61)$$

where the tranche loss at  $t \geq T_0$  defined as

$$L_t^{TR} = (L_t^{PF} - a - L_{T_0}^{PF})^+ - (L_t^{PF} - b - L_{T_0}^{PF})^+ \quad (62)$$

Eq. (62) shows that the payoff of a forward starting tranche is similar to that of a forward starting equity call spread with portfolio loss  $L_t^{PF}$  serving the role of stock price.

Valuation of the forward starting tranche reduces to calculating the expectation

$$\begin{aligned} E(L_t^{PF} - A - L_T^{PF})^+ &= E[(1 - R)(N_t^{PF} - N_T^{PF}) / N^{PF} - A]^+ \\ &= \sum_{\tilde{N}_t^C, \tilde{N}_T^C} \int P(N_t^P, \tilde{N}_t^C | N_T^P, \tilde{N}_T^C, \lambda_T) P(N_T^P, \tilde{N}_T^C, \lambda_T) (L_t^{PF} - A - L_T^{PF})^+ d\lambda_T \end{aligned} \quad (63)$$

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<sup>15</sup> For simplicity, we assume all payments are made at the payment date.

where

$$P(N_t^P, \bar{N}_t^C | N_T^P, \bar{N}_T^C, \lambda_t) = \int P(N_t^P, \bar{N}_t^C, \lambda_t | N_T^P, \bar{N}_T^C, \lambda_T) d\lambda_t \quad (64)$$

The exposition in Sec. 2.8 indicates that the expectation in (63) is different under the SI model from under the LI model. Therefore, the SI model and LI model are not equivalent for forward starting tranches.

**Remark 3.2.1:** Equity forward starting options have been priced using, in decreasing level of sophistication, SV model, LV model, and BS model with a suitably chosen implied volatility. The SI model with non-zero volatility  $\sigma(\lambda_t, N_t^P, t)$  is compared to the SV model where the default intensity plays the role of equity stochastic volatility. The LI model is the counterpart to the LV model. Jackson and Zhang (2007) proposed a factor copula models with constant correlation for tranche pricing that is analogues to the Black-Sholes model with an implied volatility for equity call option.

### 3.3. CDS Tranche Option

Option on tranche is a European option that gives the option holder the right but not obligation to buy or sell credit protection at a future time on a predetermined tranche with a specified spread (strike). Unlike the single name CDS option which ceases to exist after default, the tranche option does not knock out after a default in the portfolio. Upon exercising the option, the option buyer can settle for tranche loss occurred prior to the option expiry. The premium payment is based on the remaining tranche notional amount.

Let the option expiry be  $T_0$ , the underlying tranche maturity be  $T_N$ , and the option strike be  $S$ . The value of a payer tranche option, which gives the holder the right to buy credit protection and to pay the spread  $S$ , is equal to<sup>16</sup>

$$D_{T_0} E_0 \left\{ \sum_{j=1}^N \frac{D_{T_j}}{D_{T_0}} [E_{T_0}(L_{T_j}^{TR}) - E_{T_0}(L_{T_{j-1}}^{TR})] - S \sum_{j=1}^N \frac{D_{T_j}}{D_{T_0}} [AD - E_{T_0}(L_{T_j}^{TR})] \Delta T_j + L_{T_0}^{TR} \right\}^+ \quad (65)$$

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<sup>16</sup> To simplify notion, we assume that the tranche loss in  $\Delta T_i$  is paid out at  $T_i$  and no accrued coupon.

where the tranche loss function  $L_t^{TR}$  is defined by Eq. (58), and  $AD = DP - AP$ .  $L_{T_0}^{TR}$  is the tranche loss at the option expiry.

It is seen from Eq. (60) that the tranche option price depends on the joint default distribution at option expiry  $T_0$  and coupon payment time  $T_j$ . Therefore, a stochastic intensity model will yield a different price than the corresponding local volatility model. Examples of numerical difference in pricing of tranche options between SI model and LI models can be found in Arnsdorf and Halperin (2008) and Lopatin and Misirpashaev (2007).

#### **4. Calibration**

The model calibration is to fit the parent default intensity and the child conditional default probability function  $g_k(x_k)$  to the liquid market prices of tranches which reference either the parent or the children. We consider two fundamental situations:<sup>17</sup>

- 1) Portfolio Thinning in which the bespoke portfolio is a child of an index.
- 2) Portfolio Enlargement in which the bespoke portfolio is the parent that includes one or more indices as children.

##### **4.1. Portfolio Thinning**

When the bespoke portfolio is a subset of an index, it is natural to select the index as the parent. Using a term analogous to random thinning [Giesecke et al. 2009], we refer this case as (random) portfolio thinning in the sense that the bespoke portfolio can be viewed as a constituent of the index.

For portfolio thinning, we fit the parent default intensity  $\lambda(N_i^P, t)$  to standard tranche prices and liquid tranche option prices if appropriate. Since the parent is an index, this stage of calibration is done independent of the children. The actual calibration

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<sup>17</sup> Here we do not consider bespoke credits. In the presence of bespoke credits which should be grouped as a child, the model will also need match the average spread of this child.

procedure is model dependent. Here we do not elaborate on calibration of default intensity model to standard tranches.

Having determined  $\lambda(n,t)$ , we next calibrate the child conditional default probability  $C_k(N_t^P, \bar{N}_t^C, t)$  for all children. As stated earlier, we simultaneously solve for  $\xi_k(t), k = 1, \dots, K$ . Since the children are bespoke portfolios, they are unlikely to have liquid tranches.<sup>18</sup> Therefore, we fit the model spread for each child to the market implied intrinsic spread of that child. The child intrinsic spread for a maturity can be calculated from the constituent CDS curves in the child. This results in  $K$  equations for  $K$  unknowns.

#### 4.2. Portfolio Enlargement

The case in which the bespoke portfolio is the parent is potentially very important. From the practical standpoint, a bespoke reference portfolio often has only a portion of credits overlapping with a particular index. In this case, the first step is to construct the parent portfolio that is the union of some relevant indices and the bespoke. The indices are considered children. Now, our problem consists of a bespoke parent that contains indices as children.

When the bespoke portfolio is the parent of index children, the parent default intensity  $\lambda_t$  cannot be directly fitted to the index tranches because the parent contains more credits than do the tranche reference portfolios. In this case, we fit the multi-portfolio default model to the standard tranches referencing the (child) index portfolio. To this end, we must calibrate simultaneously the parent default intensity  $\lambda_t$  and the child conditional default probability  $C_k(N_t^P, \bar{N}_t^C, t), k = 1, \dots, K$ .

[1] Specify the parent default intensity model.

[2] Specify the child portfolio conditional default probability  $C_k(n, \bar{m}, t)$  in terms of Eqs. (12), (13) and (14).

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<sup>18</sup> If needed, we can incorporate tranche prices by best fitting to the tranches referencing children.

[3] Best fit  $\lambda_i$  and  $\xi_k(t), k = 1, \dots, K$  to the liquid tranches referencing index children and index spreads.

**Remark 4.2.1:** Bespoke portfolio often contains bespoke credits that are not constituents of any liquid index. Under our model, these bespoke credits will normally be grouped into a separate child portfolio, and the parent would contain the indices and the child of the bespoke credits. This is a case of portfolio enlargement, and hence the above calculation procedure applies.

## 5. Model Applications and Limitations

### 5.1. Bespoke Tranche Referencing Two Indexes

As stated earlier in this paper, the goal of our model is to establish relationship between the bespoke portfolio and some relevant indexes so we can incorporate the market index pricing information into the bespoke tranche pricing. In other words, we want to value the bespoke tranche in a way consistent with the pricing of standard tranches and indexes. In this section, we discuss model application through a practical example.

**Example:** A 3-7% STCDO (single-tranche CDO) referencing 100 obligors of which 50 belong to the iTraxx Europe index and the other 50 are members of CDX.NA.IG index.<sup>19</sup>

The current standard method to price such a bespoke STCDO is the one-factor copula model with some base correlation (BC) mapping. For this case, the indices to which the bespoke is mapped are naturally the CDX index and iTraxx index. Base correlation mapping can be done in several ways as described in Baheti and Morgan (2007) and Turc et al. (2006). For example, the BC for the bespoke equity tranche is the weighted average of the BCs of the index equity tranches with the same maturity and detachment point, where the weights for CDX and iTraxx are 0.5. Since 7% is not a standard iTraxx tranche detachment point, the BC for 7% DP in the iTraxx must be

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<sup>19</sup> Since CDX is traded in USD and iTraxx is traded in Euro, a currency component will be needed and the spreads in iTraxx will need to be adjusted due to the currency difference [Ehlers and Schonbucher, 2006]. We do not consider these complications. However, we note that the model does allow specification of different notional amount among children.

interpolated from the BCs for the standard iTraxx tranches. The interpolation of DP should ensure that the equity expected loss be an increasing function of the detachment point. It is well documented that BC interpolation is not guaranteed to be arbitrage-free [Schloegl et al., 2008].

We take an alternative approach to building a relation between the bespoke reference portfolio and the indices. First we enlarge the portfolio such that the enlarged portfolio (parent) contains both CDX and iTraxx indexes as children. We then shrink the parent to the bespoke portfolio which contains only a portion of the indices.

Let the bespoke portfolio be denoted by  $S_B$ , CDX index as  $S_C$ , and iTraxx index by  $S_I$ . Furthermore, let  $A = S_B \cap S_C$  and  $B = S_B \cap S_I$ . We also denote complement portfolios of  $A$  and  $B$  relative to their respective enclosing index by  $A^c = S_C \setminus A$  and  $B^c = S_I \setminus B$ .

Clearly, the parent portfolio is  $S_P = S_C \cup S_I = A \cup A^c \cup B \cup B^c$ . As discussed in section 2.9, different recovery rates may be assigned to CDX and iTraxx.

Here, we outline two calibration methods. The first method involves a two-stage computation. Each stage solves a two-child problem. The second method solves a four-child problem. The advantage of the first method is that it transform a four-child problem into two two-child problems. And two-child problem is much simpler to solve than four-child problem.

The computational procedure of method 1 is as follows:

- 1) Assume a top-down default intensity process for the “global” parent portfolio  $S_P$ . Any of the four parent models mentioned earlier may be used. The choice is largely based on convenience. At this stage, the model is still undetermined in the sense that the model parameters remain undetermined.
- 2) Assume that the children to be  $S_C$  and  $S_I$ .

- 3) Assume conditional default probability for  $S_C$  and  $S_I$ . The child conditional default probabilities are different, reflecting different riskiness of CDX and iTraxx. We also assign (different) recovery rates for  $S_C$  and  $S_I$ .
- 4) Jointly calibrate the default intensity for the portfolio and the conditional default probabilities for  $S_C$  and  $S_I$  to the standard CDX and iTraxx tranches using the procedure outlined in section 4.2.
- 5) Construct the default intensity processes for  $S_C$  and  $S_I$  using Eq. (40). This gives the default intensity for  $S_C$  and  $S_I$ . This completes the stage 1.
- 6) Take  $S_C$  as a parent with the default intensity process calculated in step 5), compute the child conditional default probability for  $A$  and  $A^C$  by solving a portfolio thinning problem involves two children using the procedure of section 4.1. The calibrating prices are the spreads of  $A$  and  $A^C$ . Repeat the same for  $S_I$  and the two children  $B$  and  $B^C$ . This is the stage 2.
- 7) Calculate the price of the tranche referencing the bespoke portfolio  $S_B$ .

At first glance, the above way to obtain the bespoke portfolio default process may seem convoluted. In our view, this is a sensible way to link the bespoke portfolio to the CDX and iTraxx indices which have substantial overlap – 40% in this case – with our bespoke portfolio where both CDX and iTraxx have 125 names. So 40% of the indexes are in the bespoke portfolio. The most important is that through this way, we obtain a dynamic default process for the bespoke portfolio which cannot be achieved by base correlation mapping. The dynamic process can then be used to consistently price and to risk manage the bespoke tranche.

The method 2 is more direct but requires solving a more complicated four-child problem.

- 1) Assume a top-down default intensity process for the parent portfolio  $S_P$ .
- 2) Divide the parent into four children,  $A, A^c, B$  and  $B^c$ .
- 3) Simultaneously calibrate the problem to standard tranches of CDS and iTraxx as well as the spreads of the four children.

### 5.2. Credit Substitution

Another potential application is the credit substitution. Managed bespoke tranche permits credit substitution where one or more credits in the bespoke portfolio are replaced with other credits of equal notional amounts. The purposes of the credit substitution can be one or a combination of the followings:

- 1) To improve or maintain the rating of the tranche.
- 2) To prevent credit deterioration.
- 3) To lock in gains on credits whose spreads have tightened substantially.

In practice, the effect of credit substitution on the tranche involves an adjustment of the tranche subordination to account for the trading gain or loss caused by the credit substitution without changing the tranche spread. The idea is that the expected loss on the tranche must remain the same before and after the substitution. When a credit with a higher spread is replaced by another credit with a lower spread, the bespoke portfolio becomes less risky and the tranche subordination must be adjusted downwards to reflect the reduced overall portfolio risk. Conversely, if a lower spread credit is replaced by a higher spread one, the subordination must be adjusted upwards to reflect the increased portfolio risk.

Under the factor copula model framework, calculation of the subordination adjustment entails first to calculate the base correlations for the new tranche using a base correlation mapping scheme, and then calculate the expected loss of the new tranche. Adjust the new tranche subordination until the post substitution tranche expected loss is

equal to that before substitution or the difference is minimized within some predetermined tolerance.

Halperin and Tomecek (2008) suggested an interesting method for the credit substitution. First, the single name credit default dynamic process consistent with the portfolio default dynamics is obtained by random thinning. Second, the default probabilities of the single names of the new portfolio are inferred, and the top-down default dynamic process of the new portfolio is updated. The expected loss of the tranche after credit substitution can be calculated from the new portfolio default process. See their paper for more details.

We assume that the pre-substitution bespoke portfolio consists of sub-portfolios  $A$  and  $B$  where  $B$  is replaced by portfolio  $C$ . In our model framework, the expected loss on the new tranche referencing portfolio  $A \cup C$  is evaluated as follows:

- 1) Construct the dynamic default processes for the overall portfolio  $A \cup B \cup C$ , and sub-portfolio  $A \cup B$  and  $A \cup C$  using the method described above.
- 2) The pre-substitution tranche expected loss is calculated based on the portfolio  $A \cup B$ .
- 3) The post substitution tranche expected loss is calculated from the portfolio  $A \cup C$ .
- 4) Adjust the subordination of the post substitution tranche until the post substitution tranche expected loss is equal to that of the pre-substitution, or the error is minimized.

### 6. Conclusions

We have presented a multi-portfolio model for the pricing of bespoke portfolio CDO tranche. A fundamental model assumption is that the bespoke portfolio has a child-parent relationship with an index or multiple indices. The bespoke portfolio may be a sub-portfolio of an index, or it may be the parent of one or several indices, or it may simply have overlapping with indices. This assumption enables to establish a relationship between the bespoke portfolio default process and the index default process. As a result, the bespoke tranche pricing is consistent with the index tranche pricing.

We have taken as given the default model for the parent portfolio. There are many choices for such a model in the open literatures. We have proposed a model for

distribution of the parent default among the children. The child conditional default probability model is very simple while satisfying all the required constraints. The default process in the children is a combination of the parent default process and the child conditional probability.

The multi-portfolio model is calibrated to the index tranches and available instruments of the bespoke portfolio. The calibration is considerably simpler when the parent portfolio is an index than when the bespoke portfolio is the parent. In the former case, bespoke default process can be built in two separate stages. First, the parent default process is calibrated to the index without regarding the child portfolio. Second, the child conditional default probability is calibrated to the available instruments of the bespoke portfolio. In the latter case where the bespoke portfolio is the parent and the indices are the children, we must build the default process for the bespoke portfolio and calibrate it to the index tranches as well as the available prices of the bespoke portfolio. In this case, the parent default process is likely the one that is of interest. Once calibrated, the model can be used to price any bespoke instruments.

Finally, we note that this paper is the methodology part of the model. It does not address the numerical implementation although major components of implementation have been outlined here and in the references cited.

### **Appendix: The Two-Child Case**

One important practical case is where there are two sub-portfolios. The bespoke portfolio can be either the parent or one of the children. In the appendix, we describe an algorithm for efficient solution of the local intensity model equation (26) for the two-child case.

When  $K = 2$ , a default must be in either child 1 or child 2. We have  $n = u + v$  and  $N = M_1 + M_2$  where  $u$  is the default level in child 1 and  $v$  is the default level in child 2. To lighten notation, we omit the dependence on  $n$  and suppress  $t$ . So  $P(n, \bar{m}, t) = P(u, v)$  and  $C_k(n, \bar{m}, t) = C_k(u, v)$  where  $k = 1, 2$ . The local intensity  $G(n, t)$  in Eq. (27) is a function of the parameters of the stochastic intensity  $\lambda_t$ .

Using the notation defined above, Eq. (26) becomes

$$\frac{dP(u, v)}{dt} = -G(n)P(u, v) + G(n-1)\{C_1(u-1, v)P(u-1, v) + C_2(u, v-1)P(u, v-1)\} \quad (\text{A.1})$$

Define  $W = (M_1 + 1)(M_2 + 1)$ , the total number of  $P(u, v, t)$ , and  $Q$  a  $W$ -dimensional vector where the component of vector  $Q$  is defined by the mapping

$$Q_{J(u, v)}(t) = P(u, v, t), \quad \text{with} \quad J(u, v) = u(M_2 + 1) + v \quad (\text{A.2})$$

subjected to the constraints

$$0 \leq u \leq M_1 \quad \text{and} \quad 0 \leq v \leq M_2 \quad (\text{A.3})$$

Using mapping (A.2), Eq. (A.1) can be rewritten as

$$\frac{dQ_j(t)}{dt} = a_{jj}(t)Q_j(t) + a_{j-1, j}(t)Q_{j-1}(t) + a_{j-(M_2+1), j}(t)Q_{j-(M_2+1)}(t) \quad (\text{A.4})$$

where  $j = J(u, v)$ ,  $j-1 = J(u, v-1)$ ,  $j-(M_2+1) = J(u-1, v)$ , and the coefficient matrix  $A = (a_{ij})_{W \times W}$  is defined by

$$\begin{aligned} a_{jj}(t) &= -G(n, t), \\ a_{j-(M_2+1), j}(t) &= G(n-1, t)C_1(u-1, v, t), \\ a_{j-1, j} &= G(n-1, t)C_2(u, v-1, t), \\ a_{ij}(t) &= 0, \quad \text{Otherwise} \quad . \end{aligned} \quad (\text{A.5})$$

The initial condition for Eq. (A.4) is

$$Q_j(0) = \delta_{j0}. \quad (\text{A.6})$$

Note that row  $j [= J(u, v)]$  of matrix  $A$  contains three non-zero entries,  $a_{jj}, a_{j, j+1}$  and  $a_{j, J(u+1, v)}$  where  $J(u, v+1) = j+1$  and  $J(u+1, v) = j + M_2 + 1$ . It is easy to verified that coefficient matrix  $A$  satisfies the property

$$a_{jj} = -\sum_{i \neq j}^W a_{ji} = -a_{j, j+1} - a_{j, j+(M_2+1)}, \quad j = 0, \dots, W-1 \quad (\text{A.7})$$

and hence is a generator matrix.

Eq. (A.4) can be rewritten conveniently in the matrix form

$$\frac{dQ}{dt} = QA \quad (\text{A.8})$$

Since the generator matrix  $A$  is deterministic, the solution of equation (A.8) is

$$Q(t) = Q(t_0) \times \text{Exp} \left( \int_{t_0}^t A(s) ds \right) \quad (\text{A.9})$$

The matrix exponential in formula (A.9) can be computed using the Pade approximation. Moler and Loan (2003) provided an overview of various methods for computing the matrix exponential.

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